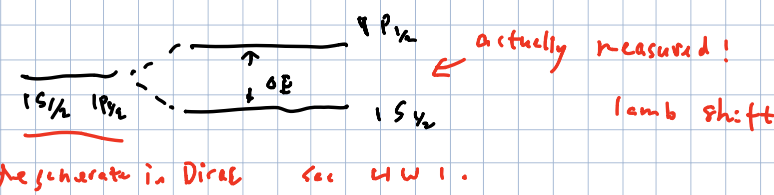
3. Lamb shift (QFT couple to Q.M.)



- Something can not be understood classically
- It's not an effect like Stark effect, when external field is turned off and disappears.
- It's really due to Quantum fluctuations

Now we couple the quantum E&M field to the hydrogen atom. or more precisely, couple to the electron in the coulomb potential.

The Hamiltonian is :

$$H = \int d^3x \; \tfrac{1}{2}\vec{\pi}_T\cdot\vec{\pi}_T + \tfrac{1}{2}(\vec{\nabla}\times\vec{A})^2 - \frac{Ze^2}{r} + \frac{(\vec{P}-e\vec{A})^2}{2m} + m_e$$

$$= \underbrace{\int d^3\vec{k}\; \omega_k a_k^\dagger a_k}_{\text{"Free photon"}} + \underbrace{\frac{\vec{p}^{\,2}}{2m} - \frac{Ze^2}{4\pi r}}_{\text{shrödinger equation, we know how to solve}} \underbrace{- e\frac{\vec{P}\cdot\vec{A}}{m} + e^2\vec{A}^2}_{\text{Interaction}} + \underbrace{m_e}_{\text{Rest energy}}$$

From $\pi_T\cdot\pi_T$

Here we ignore spin

in the Shrödinger picture:

$$\vec{A}_{free} = \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \vec{\epsilon}^r a_r e^{-i\vec{k}\cdot\vec{x}} + \vec{\epsilon}^r a_r^\dagger e^{i\vec{k}\cdot\vec{x}}$$

We want to solve the Energy Level.

We know that if $\vec{A}=0$

$$E_n = E - m_e = -\frac{1}{2}\frac{Z^2\alpha^2}{n^2}$$, and we know $\psi_n(x)$

We use perturbation theory to solve

$$H = H_0 + H_1 + H_2$$

If we consider "free" electron in the vacuum the coulomb potential drops

$$H_0 = \int d^3\vec{k}\, \omega_k a_k^\dagger a_k + \frac{p^2}{2m} - \frac{Ze^2}{r} + m_e \sim O(e^0)$$

$$H_1 = e\frac{\vec{P}\cdot\vec{A}}{2m} + \frac{e\hat{A}\hat{P}}{2m} \sim O(e)$$

$$H_2 = e^2\vec{A}^2 \sim O(e^2)$$

e is our power counting param

- Review of perturbation theory.

We want to solve $H|\phi_N\rangle = E_N|\phi_N\rangle$

To do so, we expand the operator, the Energy, the state in terms of a small power counting parameter e.

$$H = \underset{O(1)}{H^{(0)}} + \underset{O(e)}{H^{(1)}} + \underset{O(e^2)}{H^{(2)}} + \cdots + O(e^3) \cdots$$

$$|\phi_N\rangle = \underset{O(1)}{|\phi_N^{(0)}\rangle} + \underset{O(e)}{|\phi_N^{(1)}\rangle} + \underset{O(e^2)}{|\phi_N^{(2)}\rangle} + \cdots$$

$$E_N = \underset{O(1)}{E_N^{(0)}} + \underset{O(e)}{E_N^{(1)}} + \underset{O(e^2)}{E_N^{(2)}} + \cdots$$

Here we demand $H^{(0)}|\phi_N^{(0)}\rangle = E_N^{(0)}|\phi_N^{(0)}\rangle$ is an equation we can solve exactly.

and $|\phi_N^{(0)}\rangle$ form a complete base in the Hilbert space. In other words given an arbitrary state $|\Phi_K\rangle$ we can decompose $|\Phi_K\rangle$ in terms of $|\phi_N^{(0)}\rangle$

$$|\Phi_K\rangle = \sum\!\!\!\!\!\!\int_M C_{KM}|\phi_M^{(0)}\rangle$$

Also we assume $\langle\phi_{N'}^{(0)}|\phi_N^{(0)}\rangle = \delta_{N'N}$

Now we solve $H|\psi_N\rangle = E_N|\psi_N\rangle$ order by order in ϵ

$$H|\phi_N\rangle = H^{(0)}|\phi_N^{(0)}\rangle + H^{(1)}|\phi_N^{(0)}\rangle + H^{(2)}|\phi_N^{(0)}\rangle + \cdots$$
$$\underset{O(1)}{\wr} \qquad + H^{(0)}|\phi_N^{(1)}\rangle + H^{(1)}|\phi_N^{(1)}\rangle + \cdots$$
$$\underset{O(\epsilon)}{\wr} \qquad + H^{(0)}|\phi_N^{(2)}\rangle + \cdots$$
$$\underset{O(\epsilon^2)}{\wr}$$

$$= E_N^{(0)}|\phi_N^{(0)}\rangle + E_N^{(1)}|\phi_N^{(0)}\rangle + E_N^{(2)}|\phi_N^{(0)}\rangle + \cdots$$
$$\underset{O(1)}{\wr} \qquad + E_N^{(0)}|\phi_N^{(1)}\rangle + E_N^{(1)}|\phi_N^{(1)}\rangle + \cdots$$
$$\underset{O(\epsilon)}{\wr} \qquad + E_N^{(0)}|\phi_N^{(2)}\rangle + \cdots$$
$$\underset{O(\epsilon^2)}{\wr}$$

$O(1)$: $\quad H^{(0)}|\phi_N^{(0)}\rangle = E_N^{(0)}|\phi_N^{(0)}\rangle$

$$\Rightarrow \quad E_N^{(0)} = \langle\phi_N^{(0)}|H^{(0)}|\phi_N^{(0)}\rangle$$

$O(\epsilon)$: $\quad H^{(1)}|\phi_N^{(0)}\rangle + H^{(0)}|\phi_N^{(1)}\rangle = E_N^{(1)}|\phi_N^{(0)}\rangle + E_N^{(0)}|\phi_N^{(1)}\rangle$

Now we write $|\phi_N^{(1)}\rangle = \sum\!\!\!\!\!\!\int_M C_{NM}^{(1)}|\phi_M^{(0)}\rangle$

$$\Rightarrow \quad H^{(1)}|\phi_N^{(0)}\rangle + H^{(0)}\sum\!\!\!\!\!\!\int C_{NM}^{(1)}|\phi_M^{(0)}\rangle = E_N^{(1)}|\phi_N^{(0)}\rangle + E_N^{(0)}\sum\!\!\!\!\!\!\int_M C_{NM}^{(1)}|\phi_M^{(0)}\rangle$$

$\times\langle\phi_{N'}^{(0)}|$
$\Rightarrow \quad \langle\phi_{N'}^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle + E_{N'}^{(0)}C_{NN'}^{(0)} = E_N^{(1)}\delta_{NN'} + E_N^{(0)}C_{NN'}^{(1)}$

If $N'=N \quad \Rightarrow \quad E_N^{(1)} = \langle\phi_N^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle$

If $N'\neq N \quad \Rightarrow \quad C_{NN'}^{(1)} = \dfrac{\langle\phi_{N'}^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle}{E_N - E_{N'}} \quad \Rightarrow \quad |\phi_N^{(1)}\rangle = \sum\!\!\!\!\!\!\int_{M\neq N}|\phi_M^{(0)}\rangle\dfrac{\langle\phi_M^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle}{E_N - E_M}$

$C_{NN}^{(1)} = 0$

$\mathcal{O}(\epsilon^2)$:

$$H^{(2)}|\phi_N^{(0)}\rangle + H^{(1)}|\phi_N^{(1)}\rangle + H^{(0)}|\phi_N^{(2)}\rangle = E_N^{(2)}|\phi_N^{(0)}\rangle + E_N^{(1)}|\phi_N^{(1)}\rangle + E_N^{(0)}|\phi_N^{(2)}\rangle$$

$$H^{(2)}|\phi_N^{(0)}\rangle + H^{(1)} \sum\!\!\!\!\!\!\int C_{NM}^{(1)}|\phi_M^{(0)}\rangle + H^{(0)} \sum\!\!\!\!\!\!\int C_{NM}^{(2)}|\phi_M^{(0)}\rangle$$

$$= E_N^{(2)}|\phi_N^{(0)}\rangle + E_N^{(1)} \sum\!\!\!\!\!\!\int C_{NM}^{(1)}|\phi_M^{(0)}\rangle + E_N^{(0)} \sum\!\!\!\!\!\!\int C_{NM}^{(2)}|\phi_M^{(0)}\rangle$$

$\times \langle\phi_{N'}^{(0)}|$

$\Rightarrow$

$$\langle\phi_{N'}^{(0)}|H^{(2)}|\phi_N^{(0)}\rangle + \sum\!\!\!\!\!\!\int C_{NM}^{(1)}\langle\phi_{N'}^{(0)}|H^{(1)}|\phi_M^{(0)}\rangle + E_{N'}^{(0)} C_{NN'}^{(2)}$$

$$= E_N^{(2)}\delta_{NN'} + E_N^{(1)} C_{NN'}^{(1)} + E_N^{(0)} C_{NN'}^{(2)}$$

If $N'=N$, we get

$$E_N^{(2)} = \langle\phi_N^{(0)}|H^{(2)}|\phi_N^{(0)}\rangle + \sum\!\!\!\!\!\!\int C_{NM}^{(1)}\langle\phi_N^{(0)}|H^{(1)}|\phi_M^{(0)}\rangle$$

$$\Rightarrow \quad E_N^{(2)} = \langle\phi_N^{(0)}|H|\phi_N^{(0)}\rangle + \sum\!\!\!\!\!\!\int_{M\neq N} \frac{\langle\phi_N^{(0)}|H^{(1)}|\phi_M^{(0)}\rangle\langle\phi_M^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle}{E_N - E_M}$$

$$E = E_N^{(0)} + E_N^{(1)} + E_N^{(2)} + \mathcal{O}(\epsilon^3)$$

with

$$E_N^{(0)} = \langle\phi_N^{(0)}|H^{(0)}|\phi_N^{(0)}\rangle$$

$$E_N^{(1)} = \langle\phi_N^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle$$

$$E_N^{(2)} = \langle\phi_N^{(0)}|H|\phi_N^{(0)}\rangle + \sum\!\!\!\!\!\!\int_{M\neq N} \frac{\langle\phi_N^{(0)}|H^{(1)}|\phi_M^{(0)}\rangle\langle\phi_M^{(0)}|H^{(1)}|\phi_N^{(0)}\rangle}{E_N - E_M}$$

— End of review

Before solving $H|\phi_N\rangle = E_N|\psi_N\rangle$ for hydrogen.

we first consider a free electron in the vacuum.

But the vacuum is full of E&M quantum fluctuations.

$H = H_0 + H_1 + H_2$

$$H_0 \equiv \int d^3\vec{k}\, \omega_k a_k^\dagger a_k + \frac{\vec{p}^2}{2m} + m_e \qquad \sim \mathcal{O}(e^0)$$

$$H_1 = e\frac{\vec{p}\cdot\vec{A}}{2m} + \frac{e\hat{A}\cdot\hat{p}}{2m} \sim \qquad \sim \mathcal{O}(e)$$

$$H_2 = e^2 \hat{A}^2 \qquad \sim \mathcal{O}(e^2)$$

e is our power counting param

Therefore

$$|\phi_N^{(0)}\rangle = |\psi^{(0)}\rangle \cdot |0\rangle \leftarrow \text{vacuum for E\&M field.}$$

momentum eigenstate for a free electron $\langle x|\psi_k^{(0)}\rangle \propto e^{i\vec{k}\cdot\vec{x}}$

$$E_N^{(0)} = E_k^{(0)} = \langle \varphi_{\vec{p}}^{(0)} 0|\, \frac{\vec{p}^2}{2m} + \int d^3\vec{k}\, \omega_k a_k^\dagger a_k\, |\varphi_{\vec{p}}^{(0)}\, 0\rangle$$

$$= \langle \varphi_{\vec{p}}^{(0)}|\frac{\vec{p}^2}{2m}|\varphi_{\vec{p}}^{(0)}\rangle\langle 0|0\rangle + \langle \varphi_{\vec{p}}^{(0)}|\varphi_{\vec{p}}^{(0)}\rangle\langle 0|\int d^3k\, \omega_k a_k^\dagger a_k |0\rangle$$

$$= \frac{\vec{p}^2}{2m}$$

$$E_N^{(1)} = \langle \varphi_{\vec{p}}^{(0)} 0|\, \frac{e}{2m}\hat{p}\cdot\hat{A}\, |\varphi_{\vec{p}}^{(0)}\, 0\rangle$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} \langle \varphi_{\vec{p}}^{(0)}|\frac{e}{m}\hat{p}\cdot\vec{\epsilon}\, e^{i\vec{k}\cdot\vec{x}}|\varphi_{\vec{p}}^{(0)}\rangle \underbrace{\langle 0|a_k|0\rangle}_{0} + \text{c.c.}$$

$$= 0$$

$$\bar{E}_N^{(2)} = \underbrace{\langle \phi_N^{(0)} | H^{(2)} | \phi_N^{(0)} \rangle}_{①} + \underbrace{\sum_{M \neq N} \frac{\langle \phi_N^{(0)} | H^{(1)} | \phi_M^{(0)} \rangle \langle \phi_M^{(0)} | H^{(1)} | \phi_N^{(0)} \rangle}{E_N - E_M}}_{②}$$

① : $\quad |\phi_N^{(0)}\rangle = |\varphi_{\vec{p}}^{(0)}\rangle |0\rangle, \quad H^{(2)} = :e^2 \vec{A}^2:$

← in any case the vacuum energy is NOT observable here we use the normal product to set it to zero, safely

$$\Rightarrow ① = \langle \varphi_{\vec{p}}^{(0)}, 0 | :e^2 \vec{A}^2: | \varphi_{\vec{p}}^{(0)}, 0 \rangle$$

$$= \langle \varphi_{\vec{p}}^{(0)} | \varphi_{\vec{p}}^{(0)} \rangle \langle 0 | :e^2 \vec{A}^2: | 0 \rangle$$

$$\propto \langle \varphi_{\vec{p}}^{(0)} | \varphi_{\vec{p}}^{(0)} \rangle \langle 0 | :aa + a^+a + aa^+ + a^+a^+: | 0 \rangle$$

$$\propto \quad \cdots \quad \langle 0 | \; aa + 2a^+a + a^+a^+ \; | 0 \rangle \; = 0$$

② : $\quad H_1 = e \frac{\vec{p} \cdot \vec{A}}{2m} + e \frac{\vec{A} \cdot \vec{p}}{2m} \equiv \frac{e}{2m} \{ \vec{p}, \vec{A} \}$

$$\vec{A} = \int \frac{d^3k}{\sqrt{2\omega_k}} \left(\vec{\varepsilon}^r a_r e^{-i\vec{k}\cdot\vec{x}} + \vec{\varepsilon}^{*r} a_r^+ e^{i\vec{k}\cdot\vec{x}} \right)$$

We note for later use that:

$$\tfrac{1}{2} \langle \varphi_{\vec{p}}^{(0)}, 0 | \{ \vec{p}, \vec{A} \} | \varphi_{\vec{p}'}^{(0)} \, 1_r(\vec{k}) \rangle$$

$$= \int \frac{d^3 \vec{k}'}{\sqrt{2\omega_k'}} \sum_{r'} \langle 0 | a_{r'}(\vec{k}') | 1_r(\vec{k}) \rangle \langle \varphi_{\vec{p}}^{(0)} | \underbrace{\tfrac{1}{2} \{ \vec{p}, e^{i\vec{k}\cdot\vec{x}} \}}_{\equiv \; \mathbb{P}} \cdot \vec{\varepsilon}^{*r'} | \varphi_{\vec{p}'}^{(0)} \rangle$$

$$= \quad \cdots \quad \tfrac{1}{2} \langle \varphi_{\vec{p}}^{(0)} | \vec{p} \, e^{i\vec{k}\cdot\vec{x}} + e^{i\vec{k}\cdot\vec{x}} \cdot \vec{p} \, | \varphi_{\vec{p}'}^{(0)} \rangle \cdot \vec{\varepsilon}^{*r'}$$

$$= \quad \cdots \quad \tfrac{1}{2} \vec{p} \cdot \vec{\varepsilon}^{*r'} \langle \varphi_{\vec{p}}^{(0)} | e^{i\vec{k}\cdot\vec{x}} | \varphi_{\vec{p}'}^{(0)} \rangle + \langle \varphi_{\vec{p}}^{(0)} | e^{i\vec{k}\cdot\vec{x}} | \varphi_{\vec{p}'}^{(0)} \rangle \vec{p}' \cdot \vec{\varepsilon}^{*r'} \tfrac{1}{2}$$

$$= \quad \cdots \quad \tfrac{1}{2} (\vec{p} + \vec{p}') \cdot \vec{\varepsilon}^{*r'} \cdot \langle \varphi_{\vec{p}}^{(0)} | e^{i\vec{k}\cdot\vec{x}} | \varphi_{\vec{p}'}^{(0)} \rangle$$

Now we insert $\int d^3x' \, |x'\rangle\langle x'| = 1$ to find

$$\Rightarrow \langle \varphi^{(0)}_{\vec{p}} | \mathbb{P}\cdot\vec{\epsilon}^{\,*r'} | \varphi^{(0)}_{\vec{p}'}\rangle = \int d^3x' \, d^3x'' \times (\vec{p}+\vec{p}')\,\vec{\epsilon}^{\,*r'} \, \tfrac{1}{2}$$

$$\mathbb{P} = \tfrac{1}{2}\{\vec{P}, e^{i\vec{k}\cdot\vec{x}}\}$$

$$\times \langle \varphi^{(0)}_{\vec{p}} | x'\rangle\langle x' | e^{i\vec{k}\cdot\vec{x}} | x''\rangle\langle x'' | \varphi^{(0)}_{\vec{p}'}\rangle$$

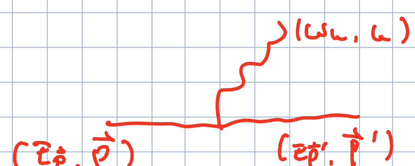
$$= \int d^3x' \, \cancel{d^3x''} \times (\vec{p}+\vec{p}')\,\vec{\epsilon}^{\,*r'} \, \tfrac{1}{2}$$

$$\times \, e^{-i\vec{p}\cdot\vec{x}'} \, e^{i\vec{k}\cdot\vec{x}'} \, \cancel{\delta^3(x'-x'')} \, e^{i\vec{p}'\cdot\vec{x}''}$$

$$= \int d^3x' \, (\vec{p}+\vec{p}')\,\vec{\epsilon}^{\,*r'} \, e^{-i(\vec{p}-\vec{k}-\vec{p}')\cdot\vec{x}'} \, \tfrac{1}{2}$$

$$= \tfrac{1}{2}(\vec{p}+\vec{p}-\vec{k})\cdot\vec{\epsilon}^{\,*r'} \cdot \delta^3(\vec{p}-\vec{k}-\vec{p}')$$

$$\underbrace{\vec{p}\cdot\vec{\epsilon}^{\,*r'}}_{\text{used } \vec{k}\cdot\vec{\epsilon}=0} \cdot \underbrace{\delta^3(\vec{p}-\vec{k}-\vec{p}')}_{\text{3-momentum conservation}}$$



Therefore

From $\vec{A}$

non-rel. normalization for Q.M.

relativistic normalization for photon

$$② = \frac{e^2}{m^2} \sum_{r'} \sum_{r''} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}} \frac{d^3\vec{k}''}{\sqrt{2\omega_{k''}}} \underbrace{\int d^3\vec{p}' \sum_{r} \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_k}}_{\sum_{m\neq n}}$$

$$\times \frac{\langle \overbrace{\varphi^{(0)}_{\vec{p}},\, 0}^{\phi_n^{(0)}} \,|\, \mathbb{P}\cdot\vec{\varepsilon}^{\,r'} a_{r'} \,|\, \overbrace{\varphi^{(0)}_{\vec{p}'},\, 1_r(\vec{k})}^{\phi_m^{(0)}} \rangle \langle \varphi^{(0)}_{\vec{p}'},\, 1_r(\vec{k}) \,|\, \mathbb{P}\cdot\vec{\varepsilon}^{\,*r''} a^{\dagger}_{r''} \,|\, \varphi^{(0)}_{\vec{p}},\, 0 \rangle}{\underbrace{E_{\vec{p}} - E_{\vec{p}'} - \omega_k}_{E_n - E_m}}$$

we pause here to see the meaning of the 2nd order perturbation theory

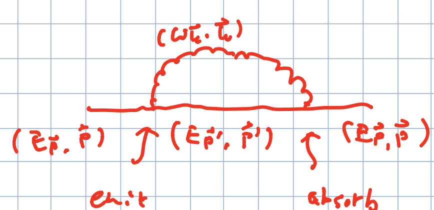
$\langle \varphi^{(0)}_{\vec{p}'}\, 1_r(\vec{k}) \,|\, \mathbb{P}\cdot\vec{\varepsilon}^{\,*r''} a^{\dagger}_{r''} \,|\, \varphi^{(0)}_{\vec{p}}\, 0 \rangle$ → probability amplitude for an electron with $\vec{p}$ emits 1 photon with momentum $\vec{k}$ and pol. r. the momentum of the electron becomes $\vec{p}'$

$a^{\dagger}_{r''}$: create a photon

$\langle \varphi^{(0)}_{\vec{p}},\, 0 \,|\, \mathbb{P}\cdot\vec{\varepsilon}^{\,r'} a_{r'} \,|\, \varphi^{(0)}_{\vec{p}'}\, 1_r(\vec{k}) \rangle$ → probability amplitude for an electron $\vec{p}'$ absorbs a photon $(\vec{k}, r)$. the momentum changed to $\vec{p}$.

$a_{r'}$: annihilate a photon

$\sum$ All possible $\vec{p}'$ and $\vec{k}$ state



In principle, for the time independent perturbation theory

$$\vec{p} = \vec{p}' + \vec{k}$$

If

$$\vec{k} \ll \vec{p} \sim \vec{p}'$$

we will expect

$$\vec{p}' = \vec{p}$$

and $\vec{p}' = \vec{p}$ state is allowed

$$= \frac{e^2}{m^2} \sum_{r} \sum_{r'} \sum_{r''} \int d^3\vec{p}\,' \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_k} \int \frac{d^3\vec{k}\,'}{\sqrt{2\omega_{k'}}} \frac{d^3\vec{k}\,''}{\sqrt{2\omega_{k''}}}$$

$$\times \frac{\langle \varphi_{\vec{p}}^{(0)} | \mathbb{P}\cdot\vec{\epsilon}^{\,r'} | \varphi_{\vec{p}\,'}^{(0)} \rangle \langle \varphi_{\vec{p}\,'}^{(0)} | \mathbb{P}\cdot\vec{\epsilon}^{\,*r''} | \varphi_{\vec{p}}^{(0)} \rangle \langle 0 | c_{r'} | 1 \rangle \langle 1 | a^{\dagger}_{r''} | 0 \rangle}{\frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}\,'^{2}}{2m} - |\vec{k}|}$$

$$= \frac{e^2}{m^2} \sum_{r} \cancel{\sum_{r'}} \cancel{\sum_{r''}} \int d^3\cancel{\vec{p}\,'} \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_k} \int \frac{d^3\cancel{\vec{k}\,'}}{\sqrt{\cancel{2\omega_{k'}}}} \int \frac{d^3\vec{k}\,''}{\sqrt{\cancel{2\omega_{k''}}}}$$

$\sim \delta^3(0) = \langle \varphi_{\vec{p}}^{(0)} | \varphi_{\vec{p}}^{(0)} \rangle$

$$\times \frac{\vec{p}\cdot\vec{\epsilon}^{\,r'}\, \delta(\vec{p} - \cancel{\vec{p}\,'} - \vec{k})\, \vec{p}\cdot\vec{\epsilon}^{\,*r''}\, \delta^3(\vec{p} - \vec{p}\,' - \vec{k})}{\frac{\vec{p}^{\,2}}{2m} - \frac{(\vec{p}-\vec{k})^2}{2m} - |\vec{k}|} \cdot \cancel{\sqrt{2\omega_{k'}}}\, \cancel{\delta_{rr'}}\, \delta(\vec{k} - \cancel{\vec{k}\,'}) \cdot \sqrt{2\cancel{\omega_{k''}}}\, \cancel{\delta_{rr''}}\, \delta(\cancel{\vec{k}} - \cancel{\vec{k}\,''})$$

$$= \frac{e^2}{m^2} \sum_{r} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2|\vec{k}|} \langle \varphi_{\vec{p}}^{(0)} | \varphi_{\vec{p}}^{(0)} \rangle$$

$$\times\; p_i\, p_j\; \epsilon_i^{r}\; \epsilon_j^{*r} \left[- \frac{2m}{k^2 - 2|\vec{p}||\vec{k}|\cos\theta + 2|\vec{k}|m} \right] \xrightarrow[p \ll m]{k \ll m} -\frac{1}{|\vec{k}|}$$

Now we note that $\vec{k} \ll m$, $\vec{p} \ll m$

$$= \frac{e^2}{m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{-2k^2} \cdot \underbrace{p_i\, p_j \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)}_{\to\, p_T^2} \langle \varphi_{\vec{p}}^{(0)} | \varphi_{\vec{p}}^{(0)} \rangle$$

transverse projector, see note before.

$$= \frac{e^2}{m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{-2k^2} \cdot p^2 \sin^2\theta\, \langle \varphi_{\vec{p}}^{(0)} | \varphi_{\vec{p}}^{(0)} \rangle$$

$$= -\frac{e^2}{m^2} \frac{p^2}{(2\pi)^3} \frac{1}{2} \int_0^{+\infty} \cancel{k^2}\, dk \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\Phi\, \frac{1}{\cancel{k^2}} \sin^2\theta$$

$$= -\frac{e^2}{(2\pi)^3} \cdot \frac{p^2}{m^2} (2\pi) \int_0^{+\infty} dk \int_{-1}^{1} d\cos\theta\, \sin^2\theta$$

$$= -\frac{e^2}{6\pi^2} \frac{p^2}{m^2} \cdot \int_0^{+\infty} dk\,.$$

Again we see infinity. This time, it is a linear divergence

Now we look back at the Hamiltonian to see what happens

$$H^{(0)} = \frac{p^2}{2m} + :\int d^3k \, a_k^\dagger a_k \omega_k :$$

but m here is a parameter. It's value is NOT predictable in our current theory. The bare value could be different from its physical value.

$$m_{bare} = m_{phys.} + \delta m$$

$$\Rightarrow \quad H^{(0)} = \frac{p^2}{2m_{bare}} = \frac{p^2}{2(m_{phys} + \delta m)}$$

$$= \frac{p^2}{2m_{phys}} - \underbrace{\frac{p^2}{2m_{phys}^2} \cdot \delta m}_{\text{move this to } H^{(1)}} + O(\delta m^2)$$

$$\Rightarrow \quad E_N^{(2)} = \langle \phi_N^{(0)} | H^{(1)} | \phi_N^{(0)} \rangle$$

$$= -\frac{e^2}{6\pi^2} \frac{p^2}{m^2} \cdot \int_0^{+\infty} dk - \frac{p^2}{2m_{phys}^2} \delta m$$

a counter term due to the bare mass parameter

$$\text{Now} \quad E = E_N^{(0)} + \cancel{E_N^{(1)}}^{\,0} + E_N^{(2)}$$

$$E_{free} = \frac{p^2}{2m_{phys}} - \frac{p^2}{2m_{phys}^2} \delta m - \frac{e^2}{3\pi^2} \frac{p^2}{2m_{phys}^2} \int_0^{\Lambda} dk \, . + O(\delta m^2)$$

a regulator for UV

by m_{phys} we mean that if we measure E_{free}, and we measure p, we have $m_{phys} \equiv \frac{p^2}{2E_{free}}$

Λ — is the scale where the theory is expected to break down

$$\Rightarrow \quad \delta m = 0 - \frac{e^2}{3\pi^2} \cdot \int_0^{\Lambda} dk$$

$$\Rightarrow \qquad \delta m = -\frac{e^2}{3\pi^2}\Lambda + \mathcal{O}(e^3)$$

$\Lambda \lesssim m_{phys}$ or $\sim \vec{p}$

- so you see that Λ is NOT necessarily very large.
  it can be a very small quantity.
  It all depends on where your theory starts to break down.
  and Λ is indeed much larger than the typical scales in the theory, e.g. here $\Lambda \sim \underset{\text{or } \vec{p}}{m_e} \gg \frac{\vec{p}^2}{2m} \sim \vec{k}$

- So Λ is really "infinity" for a person lives at scale $\sim \frac{\vec{p}^2}{2m} \sim \vec{k}$

- we don't predict mass. we fix by compare with experiments!

Now we have fixed $\delta m = -\frac{e^2}{3\pi^2}\Lambda$ at $\mathcal{O}(e^2)$ and we turn to lamb shift for Hydrogen. $\quad m\alpha^2,\ m\alpha,\ m,\ m_p$

The Hamiltonian is given by

$$H = H^{(0)} + H^{(1)} + H^{(2)}$$

$$H^{(0)} = \underbrace{\int d^3\vec{k}\ \omega_k a_k^\dagger a_k}_{\text{"free photon"}} + \underbrace{\frac{\vec{p}^2}{2m}}_{\text{"free electron"}} - \underbrace{\frac{Ze^2}{4\pi r}}_{\text{Coulomb potential}} \qquad \sim \mathcal{O}(e^0)$$

$$H^{(1)} = \frac{e\,\vec{p}\cdot\vec{A}}{2m} + \frac{e\,\vec{A}\cdot\vec{p}}{2m} \rightarrow e\,\frac{\vec{p}\cdot\vec{A}}{m} \qquad \sim \mathcal{O}(e)$$

$$H^{(2)} = e^2\vec{A}^2 - \frac{\vec{p}^2}{2m^2}\delta m \hookrightarrow \text{mass counter term} \qquad \sim \mathcal{O}(e^2)$$

Again e is our power counting parameter everything is the same, except we have coulomb potential

therefore $|\varphi_p^{(0)}\rangle \to |\varphi_n^{(0)}\rangle$, $\frac{p^2}{2m} \to E_n$

we repeat what we have done for the "free" electron.

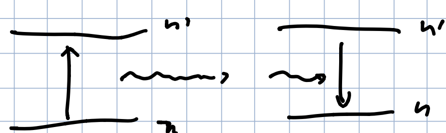
$$E_n^{(0)} = \langle \varphi_n^{(0)} 0 | H^{(0)} | \varphi_n^{(0)} 0\rangle = -\frac{1}{2} m \frac{Z^2\alpha^2}{n^2} - \frac{\vec{p}^2}{2m^2}\delta m$$

$$E_n^{(1)} = \langle\varphi_n^{(0)}0|H^{(1)}|\varphi_n^{(0)}0\rangle \propto \langle 0|a^\dagger + a|0\rangle = 0$$

$$E_n^{(2)} = -\frac{1}{2m^2}\delta m \langle \varphi_n^{(0)}|p^2|\varphi_n^{(0)}\rangle$$

$$+ \frac{e^2}{m^2}\sum_{r',r''}\int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}}\frac{d^3\vec{k}''}{\sqrt{2\omega_{k''}}} \sum_{n'}\sum_r\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}$$

$$\times \frac{\langle\varphi_n^{(0)},0|\vec{p}\cdot\vec{\epsilon}^{\,r'} a_{r'}|\varphi_{n'}^{(0)}, 1_r(k)\rangle\langle\varphi_{n'}^{(0)},1_r(k)|\vec{p}\cdot\vec{\epsilon}^{\,*r''}a^\dagger_{r''}|\varphi_n^{(0)},0\rangle}{E_n - E_{n'} - \omega_k}$$



$\Rightarrow \omega_k \sim k \sim E_n - E_{n'} \sim m Z^2\alpha^2$

$x \sim \frac{1}{p} \sim \frac{1}{Zm\alpha}$

$\Rightarrow e^{ik\cdot x} \approx e^{i m Z^2\alpha^2 \frac{1}{Z\alpha m}} \sim e^{iZ\alpha} \to 1$, $\quad Z\alpha \to 0$

justified the dipole approximation.

$$E_n^{(2)} = -\frac{1}{2m^2}\delta m \langle \varphi_n^{(0)}|p^2|\varphi_n^{(0)}\rangle$$

$$+ \frac{e^2}{m^2}\cancel{\sum_{r',r''}}\int \frac{\cancel{d^3\vec{k}'}}{\cancel{\sqrt{2\omega_{k'}}}}\frac{\cancel{d^3\vec{k}''}}{\cancel{\sqrt{2\omega_{k''}}}} \sum_{n'}\sum_r\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}$$

$$\times \frac{\langle\varphi_n^{(0)}|\vec{p}\cdot\vec{\epsilon}^{\,r'}|\varphi_{n'}^{(0)}\rangle\langle\varphi_{n'}^{(0)}|\vec{p}\cdot\vec{\epsilon}^{\,*r''}|\varphi_n^{(0)}\rangle}{E_n-E_{n'}-\omega_k} \cancel{\sqrt{2\omega_{k'}}}\,\cancel{\delta_{rr'}}\,\delta^3(\cancel{\vec{k}-\vec{k}'})\,\cancel{\sqrt{2\omega_{k''}}}\,\cancel{\delta_{rr''}}\,\delta(\cancel{\vec{k}-\vec{k}''})$$

$\to \delta_{rr'}$

$$E_n^{(2)} = -\frac{1}{2m^2}\delta m \langle \varphi_n^{(0)}|p^2|\varphi_n^{(0)}\rangle + \frac{e^2}{m^2}\sum_{n'}\int\frac{d^3k}{(2\pi)^3 2\omega_k}$$

$$\times \frac{\langle\varphi_n^{(0)}|\vec{p}_i|\varphi_{n'}^{(0)}\rangle\langle\varphi_{n'}^{(0)}|\vec{p}_j|\varphi_n^{(0)}\rangle}{E_n - E_{n'} - \omega_k}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$

$$= -\frac{1}{2m^2}\delta m \langle\varphi_n^{(0)}|p^2|\varphi_n^{(0)}\rangle + \frac{e^2}{m^2}\sum_{n'}\int_0^{\Lambda}\frac{dk}{(2\pi)^3}\cdot\frac{k}{2}\int_{-1}^{1}dc_\theta\int_0^{2\pi}d\phi\cdot(\sin^2\theta)$$

$$\times\frac{\langle\varphi_n^{(0)}|\hat{p}|\varphi_{n'}^{(0)}\rangle\langle\varphi_{n'}^{(0)}|\hat{p}|\varphi_n^{(0)}\rangle}{E_n - E_{n'} - k}$$

$$= -\frac{1}{2m^2}\delta m\langle\varphi_n^{(0)}|p^2|\varphi_n^{(0)}\rangle + \frac{1}{m^2}\frac{e^2}{6\pi^2}\sum_{n'}\int_0^{\Lambda}dk\,k$$

$$\times\frac{\langle\varphi_n^{(0)}|\hat{p}|\varphi_{n'}^{(0)}\rangle\langle\varphi_{n'}^{(0)}|\hat{p}|\varphi_n^{(0)}\rangle}{E_n - E_{n'} - k}$$

$$= \sum_{n'} -\frac{1}{2m^2}\left(-\frac{1}{3}\frac{e^2}{\pi^2}\right)\langle\varphi_n^{(0)}|p|\varphi_{n'}^{(0)}\rangle\langle\varphi_{n'}^{(0)}|p|\varphi_n^{(0)}\rangle$$

$$\times\left\{\Lambda + \int_0^{\Lambda}dk\frac{k}{E_n - E_{n'} - k} - 1 + 1\right\}$$

Linearly divergent

$$= \sum_{n'}\frac{1}{2m^2}\left(\frac{1}{3}\frac{e^2}{\pi^2}\right)\langle\varphi_n^{(0)}|p|\varphi_{n'}^{(0)}\rangle\langle\varphi_{n'}^{(0)}|p|\varphi_n^{(0)}\rangle$$

$$\times\int_0^{\Lambda}dk\ \mathrm{P.V.}\ \frac{E_n - E_{n'}}{E_n - E_{n'} - k}$$

Principle value

Logarithmic divergent.

$$E_n^{(2)} = -\sum_{n'} \frac{1}{2m^2}\frac{1}{3}\frac{e^2}{\pi^2} \langle \varphi_n^{(0)} | p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle (E_n - E_{n'})$$

$$\times \log \frac{\Lambda}{|E_n - E_{n'}|}$$

Now define

$$\langle \varphi_n^{(0)} | p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle (E_n - E_{n'}) \log \frac{\Lambda}{|E_n - E_{n'}|},$$

$$\equiv \langle \varphi_n^{(0)} | p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle (E_n - E_{n'}) \log \frac{\Lambda}{\langle E_n - E \rangle_{ave}}$$

Independent of $E_{n'}$

which allows us to solve for

$$\log \frac{1}{\langle E_n - E \rangle_{ave}}$$

$$\Rightarrow \quad E_n^{(2)} = \frac{1}{2m^2}\frac{1}{3}\frac{e^2}{\pi^2} \log \frac{\Lambda}{\langle E_n - E \rangle_{ave}} \times \sum_{n'} \langle \varphi_n^{(0)} | p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle (E_{n'} - E_n)$$

$$= \frac{1}{2m^2}\frac{1}{3}\frac{e^2}{\pi^2} \log \frac{\Lambda}{\langle E_n - E \rangle_{ave}} \cdot \frac{1}{2} \langle \varphi_n^{(0)} | [p, [H^{(0)}, p]] | \varphi_n^{(0)} \rangle \; *$$

$$= \frac{1}{2m^2}\frac{4\pi\alpha}{3\pi^2} \log \frac{\Lambda}{\langle E_n - E \rangle_{ave}} \cdot \frac{Z\alpha}{2} |\varphi_n(0)|^2 \; *_e$$

$$= \frac{Z\alpha^2}{3\pi} \frac{1}{m^2} |\varphi_n(0)|^2 \log \frac{\Lambda}{\langle E - E_n \rangle_{ave}}$$

Note if $l \neq 0$ $\varphi_n(0) = 0$

Λ dependent!

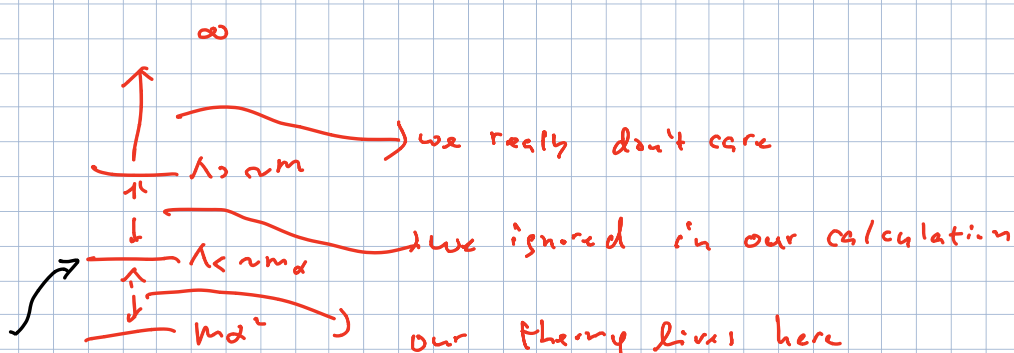
but only logarithmically.

Sensitive to UV, but not that much.

- Our prediction depends on an artificial Λ, which should not be, since $E_n^{(2)}$ is physical. This means our prediction is INCOMPLETE!.

- Indeed, we only calculated $\omega_n \sim m\alpha^2$ contributions. There're $\omega_n \sim m\alpha$ contributions we omit here. For instance, corrections to the charge e. If we add them up, this Λ dependence will be away.

- To predict Lamb shift, we need physics including all $m\alpha$, $m\alpha^2$ energy scales.



a boundary to separate 2-theories
one with typical scale $\sim m\alpha$
one with typical scale $\sim m\alpha^2$
but it is quite artificial
I can choose $2m\alpha$, $3m\alpha$, $0.5m\alpha$ ...

$m\alpha$ is a UV cutoff for our theory
but it is an IR cutoff for the theory lives in a higher scale.
And when we add these two theory the artificial cutoff should go away.

– So $E_n^{(2)} = E_{n<}^{(2)} + E_{n>}^{(2)}$

$$= \frac{Z\alpha^2}{3\pi}\frac{1}{m^2}|\psi_n(0)|^2 \log\frac{\Lambda_<}{\langle E-E_n\rangle_{avg}} + E_{n>}^{(2)}$$

in $E_{n<}^{(2)}$. $\Lambda_< \sim m\alpha$ → consistent with dipole expansion

However we know that $E_{n>}^{(2)}$ should cancel the $\log\Lambda$ dependence in $E_{n<}^{(2)}$. therefore in $E_{n>}^{(2)}$ there will be $\log\frac{\Lambda_>}{\Lambda_<}$ term. $\Lambda_> \sim m$, $\Lambda_< \sim m\alpha$

$\Lambda_>$ dependence will be reduced or eliminated by the renormalization procedure in the "full Theory" (called QED) and $\Lambda_<$ will be replaced by m_e

when add up we have

$$E_n^{(2)} = \frac{Z\alpha^2}{3\pi}\frac{1}{m^2}|\psi_n(0)|^2 \log\frac{m}{\langle E-E_n\rangle_{avg}} + \cdots$$

This is what in Bethe's original paper in which he sets Λ directly to m, arguing that the theory breaks down at $\Lambda \sim m$. this is NOT consistent with the dipole Approximation, in which we used $\omega_k \sim k \sim m\alpha^2 \ll m\alpha$.

The real logic to set $\Lambda \sim m$ is really described above.

$*$

$$\sum_{n'} \langle \varphi_n^{(0)} | p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle (E_{n'} - E_n)$$

$$= \frac{1}{2}\Big\{ \sum_{n'} \langle \varphi_n^{(0)} | p E_{n'} | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle + \langle \varphi_n^{(0)} | p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | E_{n'} p | \varphi_n^{(0)} \rangle$$

$$- \langle \varphi_n^{(0)} | E_n p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p | \varphi_n^{(0)} \rangle - \langle \varphi_n^{(0)} p | \varphi_{n'}^{(0)} \rangle \langle \varphi_{n'}^{(0)} | p E_n | \varphi_n^{(0)} \rangle \Big\}$$

$$= \frac{1}{2} \langle \varphi_n^{(0)} | p H^{(0)} p + p H^{(0)} p - H^{(0)} p p - p p H^{(0)} | \varphi_n^{(0)} \rangle$$

$$= \frac{1}{2} \langle \varphi_n^{(0)} | p [H^{(0)}, p] - [H^{(0)}, p] p | \varphi_n^{(0)} \rangle = \frac{1}{2} \langle \varphi_n^{(0)} | [p, [H^{(0)}, p]] | \varphi_n^{(0)} \rangle$$

$**$

$$\frac{1}{2} \langle \varphi_n^{(0)} | [\hat{p}, [H^{(0)}, \hat{p}]] | \varphi_n^{(0)} \rangle$$

$$= \frac{1}{2} \langle \varphi_n^{(0)} | [\hat{p}, [-\frac{Z\alpha}{|x|}, \hat{p}]] | \varphi_n^{(0)} \rangle$$

$$= \frac{1}{2} \langle \varphi_n^{(0)} | [\hat{p}, (\hat{p} \frac{Z\alpha}{|x|})] | \varphi_n^{(0)} \rangle$$

acts only inside $(\cdots)$

$$= \frac{1}{2} \langle \varphi_n^{(0)} | (\hat{p}^2 \frac{Z\alpha}{|x|}) | \varphi_n^{(0)} \rangle$$

$$= \frac{1}{2} \int d^3x' \, d^3x'' \, \langle \varphi_n^{(0)} | x' \rangle \langle x' | (\hat{p}^2 \frac{Z\alpha}{|x|}) | x'' \rangle \langle x'' | \varphi_n^{(0)} \rangle$$

$$= \frac{1}{2} \int d^3x' \, d^3x'' \, \varphi_n^{(0)}(x') \, \varphi_n^{*(0)}(x'') \, \delta^3(x' - x'') \left(-\vec{\nabla}^2 \frac{Z\alpha}{|x'|} \right)$$

$$= \frac{1}{2} \int \cancel{d^3x'} \, \cancel{d^3x''} \, \varphi_n^{(0)}(x') \varphi_n^{*(0)}(x'') \, \delta^3(\cancel{x'} - x'') \, \delta^3(\cancel{x'}) \, Z\alpha$$

$$= \frac{1}{2} Z\alpha \, |\varphi_n^{(0)}(0)|^2$$

– end of QFT part I