let's pause here to comment what we have done.

- we have quantized the E&M field within the Coulomb gauge: $\vec{\nabla}\cdot\vec{A}$

- more specifically, we have quantized only the transverse part of A^μ. i.e. $\vec{A}_1$ and $\vec{A}_2$. using

$$[A_i(\vec{x}), \Pi_j(\vec{y})] = -i\delta_{ij}^{Tr}\,\delta^3(\vec{x}-\vec{y}).$$

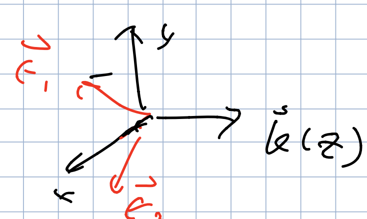
$$\delta_{ij}^{Tr} \doteq \left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)$$ $\rightsquigarrow$ a projection operator.

- For the longitudinal part:

A_0 is not dynamical. $\nabla^2 A_0 = -\rho$

$\nabla^2 A_0 = 0 \Rightarrow A_0 = 0$ for free E&M.

$\vec{A}_3$ is removed by the Gauge condition $\vec{\nabla}\cdot\vec{A} = 0$.



- The hamiltonian can be written as.

$$H = \int d^3x \quad \frac{1}{2}\dot{\pi}\cdot\dot{\pi} + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 - A_0(\vec{\nabla}\cdot\vec{E}) \qquad \pi_i = -(\partial_0 A_i + \partial_i A_0)$$

$$= \int d^3x \quad \frac{1}{2}\dot{\pi}_L\cdot\dot{\pi}_L + \frac{1}{2}\dot{\pi}_T\cdot\dot{\pi}_T + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 - A_0(\vec{\nabla}\cdot\vec{E})$$

$$\pi_T\cdot\pi_L = 0 \quad , \quad \pi_i^T \equiv \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\pi_j \qquad \vec{\pi}^L = \vec{\pi} - \vec{\pi}^T$$

$$\pi_i^T = -\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)(\partial_0 A_j + \partial_j A_0)$$

$$= -\left(\partial_0 A_i - \partial_0\frac{\partial_i\partial_j}{\partial^2}A_j\right) - (\partial_i A_0 - \partial_i A_0)$$

$$= -\partial_0\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)A_j = -\partial_0 A_j \quad \text{in Coulomb gauge}$$

$$\pi_i^L = -\partial_i A_0 - \frac{\partial_i\partial_j}{\partial^2}\partial_0 A_j = -\partial_i A_0 \quad \text{in Coulomb gauge}$$

$$A_T = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)A_j = A_i \quad \text{in Coulomb gauge.} \quad (\vec{\nabla}\cdot\vec{A} = 0)$$

so $\vec{A}$ is transverse! $\vec{A} = \vec{A}_T$. $\vec{A}_L = 0$

- for free E&M fields.

$$\pi_i^L = -\partial_i A_0 = 0. \qquad \text{since } A_0 = 0^{*}$$

$*$ the equation of motion gives

$$\vec{\nabla}^2 A_0 = 0 \Rightarrow A_0 = 0. \Rightarrow \pi_i^L = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0$$

therefor

$$H_{free} = \int d^3x \; \frac{1}{2}\vec{\pi}_T\cdot\vec{\pi}_T + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2$$

is the Hamiltonian for free E&M field, and only transverse degrees of freedom contribute. which reflects the fact that E&M waves have only 2-transverse polarizations.

And we quantized using $[A_i, \pi_j] = i\delta_{ij} \cdots$

- for interacting fields. (when we have charges) we have

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{E\&M field}} \underbrace{- A_\mu j^\mu}_{\text{Interaction}}$$

$\leftarrow$ see your HW for 1 example

Interaction j^μ can be external source or fields

$$H = \int d^3x \; \frac{1}{2}\vec{\pi}_T\cdot\vec{\pi}_T + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}\vec{\pi}_L\cdot\vec{\pi}_L - A_0(\vec{\nabla}\cdot\vec{E}) + A_\mu j^\mu$$

$$\vec{\nabla}^2 A_0 = -\rho \Rightarrow A_0(x) = -\frac{1}{\vec{\nabla}^2}\rho \Rightarrow \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot(\dot{\vec{A}} + \vec{\nabla}A_0) = \rho$$

we still have $\vec{\nabla}\cdot\vec{A} = 0$ since it's our gauge choice

therefore

$$H = \int d^3x \; \underbrace{\frac{1}{2}\vec{\pi}_T\cdot\vec{\pi}_T + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2}_{\text{"free"}} + \underbrace{\frac{1}{2}\vec{\pi}_L\cdot\vec{\pi}_L - \vec{A}\cdot\vec{j}}_{\text{Interaction}} \cancel{-A_0\rho} + \cancel{A_0}\rho$$

recall that in Coulomb gauge

$$\pi_L = -\vec{\nabla} A_0$$

and $\qquad\qquad\qquad\qquad \Rightarrow \quad \pi_L = \vec{\nabla}\frac{1}{\vec{\nabla}^2}\rho$

$$\vec{\nabla}^2 A_0 = -\rho$$

and

$$H_{int} = \int d^3\vec{x}\ \frac{1}{2}\vec{\pi}_L\cdot\vec{\pi}_L - \vec{A}\cdot\vec{j} = \int d^3\vec{x}\ \frac{1}{2}\left(\vec{\nabla}\frac{1}{\vec{\nabla}^2}\rho\right)^2 - \vec{A}\cdot\vec{j}$$

$$= \frac{1}{2}\int d^3\vec{x}\, d^3\vec{x}'\ \underbrace{\frac{1}{4\pi}\rho(\vec{x})\frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}')}_{} {}^{*} - \vec{A}\cdot\vec{j}$$

Instanteous Interaction.
No time dependence due to the gauge we choose.

Hence $\quad H = \int d^3x\ \frac{1}{2}\vec{\pi}_T\cdot\vec{\pi}_T + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2$

$$+\frac{1}{2}\int d^3x\, d^3\vec{x}'\ \frac{1}{4\pi}\rho(\vec{x})\frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}') - \int d^3\vec{x}\ \vec{A}\cdot\vec{j}$$

related to ρ $\qquad$ removed by $\vec{\nabla}\cdot\vec{A}$

We see that we have treated the longitudinal part $(A_0, \vec{\pi}_L, \vec{A}_L)$ and the transverse part $(\vec{A}_T, \vec{\pi}_T)$ differently in Coulomb gauge,
dynamical field
which making the Lorentz covariance less obvious!

It can be avoided if we choose covariant Lorenz gauge
$(\partial_\mu A^\mu = 0)$

* To see this, we consider

$$\int d^3\vec{x}\left(\vec\nabla \frac{1}{\vec\nabla^2}\rho(x)\right)\cdot\left(\vec\nabla\frac{1}{\vec\nabla^2}\rho(x)\right)$$

$$= \int d^3\vec{x}\left(\vec\nabla\frac{1}{\vec\nabla^2}\int d^3k\,\bar\rho(k)e^{i\vec k\cdot\vec x}\right)\cdot\left(\vec\nabla\cdot\frac{1}{\vec\nabla^2}\int d^3k'\,\bar\rho(k')e^{i\vec k'\vec x}\right)$$

$$= -\int d^3\vec x\int d^3\vec k\, d^3\vec k'\ \frac{\vec k}{\vec k^2}\bar\rho(k)\ \frac{\vec k'}{\vec k'^2}\bar\rho(k')\ e^{i(\vec k+\vec k')\vec x}$$

$$= -\int d^3\vec k\int d^3\vec k'\ \frac{\vec k\cdot\vec k'}{\vec k^2\vec k'^2}\ \bar\rho(k)\bar\rho(k')\ \delta^3(\vec k+\vec k')$$

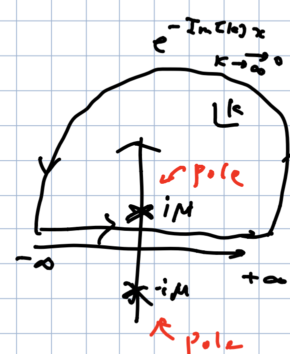
$$= \int d^3\vec k\ \frac{\cancel{\vec k^2}}{\cancel{\vec k^2}\,\vec k^2}\ \bar\rho(k)\bar\rho(-k)$$

$$= \int d^3\vec k\int d^3\vec x\, d^3\vec x'\, d^3\vec y\ \frac{1}{4\pi}\frac{1}{|\vec y|}\ e^{-i\vec k\cdot\vec y}\ \rho(\vec x)e^{-i\vec k\cdot\vec x}\rho(\vec x')e^{i\vec k\cdot\vec x'}$$

$$= \int d^3\vec k\int d^3\vec x\,d^3\vec x'\,d^3\vec y\ \frac{1}{4\pi}\frac{1}{|\vec y|}\rho(\vec x)\rho(\vec x')\ e^{-i\vec k\cdot(\vec y-\vec x'+\vec x)}$$

$$= \int d^3\vec x\,d^3\vec x'\,d^3\vec y\ \frac{1}{4\pi}\frac{1}{|\vec y|}\rho(\vec x)\rho(\vec x')\ \delta(\vec y-(\vec x-\vec x'))$$

$$= \int d^3\vec x\,d^3\vec x'\ \frac{1}{4\pi}\rho(\vec x)\frac{1}{|\vec x-\vec x'|}\rho(\vec x')$$



$$\int\frac{d^3\vec k}{(2\pi)^3}\frac{1}{\vec k^2+\mu^2}e^{i\vec k\cdot\vec x} = \frac{1}{(2\pi)^3}\int_0^{+\infty}dk\int_{-1}^{1}d\cos\int_0^{2\pi}d\Phi\ \frac{k^2}{k^2+\mu^2}\ e^{ik\cdot x\cdot\cos}$$

$$= \frac{1}{ix}\frac{1}{(2\pi)^2}\int_0^{+\infty}dk\ \frac{k}{k^2+\mu^2}\left[e^{ikx}-e^{-ikx}\right] = \frac{1}{4\pi^2}\frac{1}{ix}\int_{-\infty}^{+\infty}dk\ \frac{k\,e^{ikx}}{(k+i\mu)(k-i\mu)}$$

$$= \frac{1}{ix}\ \frac{i}{2\pi}\ \frac{1}{2\pi i}\oint dk\ \frac{k\,e^{ikx}}{(k+i\mu)(k-i\mu)} = \frac{1}{2\pi}\frac{\cancel{i}}{\cancel{i}x}\cdot\frac{\cancel{i\mu}}{2\cancel{i\mu}}\,e^{-\mu x} = \frac{1}{4\pi}\frac{e^{-\mu x}}{x}\xrightarrow{\mu\to 0}\frac{1}{4\pi}\frac{1}{x}.$$

- we introduce the fock space

$a_r(\vec{k})$ $a_r^{+}(\vec{k})$

$[a_r\ a_{r'}] = [a_r^{+}, a_{r'}^{+}] = 0$

$[a_r(k), a_{r'}^{+}(k')] = \delta_{rr'}\delta^3(\vec{k}-\vec{k}')$

$|0\rangle$ satisfies. $a_r(\vec{k})|0\rangle = 0$

we can build space for arbitrary number of photons.

$a_r^{+}(\vec{k})|0\rangle = \frac{1}{\sqrt{2\omega_k}} |1_r(\vec{k})\rangle$

1- photon with helicity r & momentum $\vec{k}$.

$$\Rightarrow \frac{1}{\sqrt{n_{r_1}!}}\{a_{r_1}^{+}(\vec{k}_1)\}^{n_{r_1}} \frac{1}{\sqrt{n_{r_2}}} [a_{r_2}^{+}(\vec{k}_2)]^{n_{r_2}} \cdots |0\rangle$$

$$= |n_{r_1}(k_1), n_{r_2}(k_2) \cdots\rangle$$

n_{r_i} photon with helicity r_i & momentum $\vec{k}_i$

– we calculate the Hamiltonian in terms of a_r $a_r^\dagger$

$$H_{free} = \sum_r \int d^3\vec{k}\, \underbrace{\left(a_r^\dagger(k)\, a_r(k) + \frac{1}{2}\delta_{rr}\,\delta(\vec{0}) \right)}\, \omega_k$$

sum of infinite harmonic oscillators
each one with $\omega = \omega_k$

1. The meaning of the Vacuum Energy.

The origin of the vacuum energy:

$$\vec{E} = -\dot{\vec{A}} \quad \propto \quad A^-(\vec{x}) - A^+(\vec{x}) \quad \propto \quad a - a^\dagger$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \propto \quad A^-(\vec{x}) + A^+(\vec{x}) \quad \propto \quad a + a^\dagger$$

$$\Rightarrow \quad \langle \vec{E} \rangle \equiv \langle 0 | \vec{E} | 0 \rangle \propto \langle 0 | a + a^\dagger | 0 \rangle = 0$$

$$\langle \vec{B} \rangle \equiv \langle 0 | \vec{B} | 0 \rangle \propto \langle 0 | a + a^\dagger | 0 \rangle = 0$$

the expectation is 0. No physical photon / field in the vacuum.

However

$$\langle \delta \vec{E} \rangle^2 \equiv \langle \vec{E}^2 \rangle - \underbrace{\langle \vec{E} \rangle^2}_{=0} = \langle 0 | \vec{E}^2 | 0 \rangle \neq 0$$

↳ calculated before

and

$$\langle \delta \vec{B} \rangle^2 \equiv \langle \vec{B}^2 \rangle - \underbrace{\langle \vec{B} \rangle^2}_{=0} = \langle 0 | \vec{B}^2 | 0 \rangle \neq 0$$

The deviation or the fluctuation is NOT 0.

even in the vacuum!!

$$\langle 0 | H_{free} | 0 \rangle = \frac{1}{2} \int d^3\vec{x} \, \langle 0 | \vec{E}^2 + \vec{B}^2 | 0 \rangle$$

$$= \frac{1}{2} \int d^3\vec{x} \left(\langle \vec{E} \rangle^2 + \langle \vec{B} \rangle^2 \right)$$

⇝ vacuum energy is due to the field fluctuation.

$$\left. \begin{array}{l} \langle 0 | \vec{E}^2 | 0 \rangle \\ \langle 0 | \vec{B}^2 | 0 \rangle \end{array} \right\} \subset \langle 0 | A^-(x) A^+(x) | 0 \rangle$$

create a photon at $\vec{x}$

immediately annihilate the photon.

So vacuum is NOT "empty".

Infinities:

We wrote before that $\langle 0 | H_{free} | 0 \rangle$ is both IR & UV infinite:

$$\langle 0 | H_{free} | 0 \rangle = 2 \times \frac{1}{2} \int d^3\vec{k} \, \omega_{\vec{k}} \, \delta(0)$$

IR infinity due to infinite space volume.

Since $\delta(0) = \int d^3\vec{x} \, e^{i\vec{x}\cdot\vec{k}} \Big|_{\vec{k}=0} = \int d^3\vec{x} = \text{Vol}.$

Define the Energy density, we can get rid of the IR divergence.

$$\mathcal{E} = \frac{\langle 0 | H_{free} | 0 \rangle}{Vol.} = 2 \times \frac{1}{2} \int d^3\vec{k}\, \omega_{\vec{k}}$$

$$= 2 \times \frac{1}{2} \int d^3\vec{k}\, |\vec{k}|$$

$$= 2 \times \frac{1}{2} \int_0^{+\infty} dk\, d\Omega_k\, |\vec{k}|^3$$

$$= 2 \times \frac{1}{2} \times 4\pi \int_0^{+\infty} dk\, k^3$$

we still have the UV divergence due to the modes with $k \to \infty$.

Solution:

1. Normal Product:

$$: a\, a^\dagger\, a\, a\, a^\dagger a^\dagger a^\dagger a^\dagger a \cdots : = a^\dagger \cdots a^\dagger \cdots a \cdots a$$

e.g. $\langle 0 | : H : | 0 \rangle = \frac{1}{2} \int d^3k\, \omega \langle 0 | : a^\dagger a + a a^\dagger : | 0 \rangle$

$$= \int d^3k\, \omega \langle 0 | a^\dagger a | 0 \rangle = 0$$

no vac energy

2. Renormalization procedure

Bare QFT is extremely sensitive to short distance physics which is not good. It means we can predict nothing without knowing the underlying theory in advance. If this is the case, WE CAN NOT EVEN DO PHYSICS.

But we said that QFT does not necessarily the fundamental theory of our nature. It may break down at some large scale Λ (or small distance of $\frac{1}{\Lambda}$).

So we have to cut off our theory at Λ or below.

Now, suppose that we have a FULL Theory which is valid to arbitrary large scale and can predict ξ

not necessarily field theory!

$$\xi = \int_0^{\infty} dk \left(\text{FULL Theory prediction} \right)$$

valid for all k different DoF. as we know today!

when $k < \Lambda$, we know that Full Theory $\longrightarrow$ QFT

→ means we can do experimental test to see QFT agrees with experimental results.

or QFT is an IR effective theory of the full theory.

Therefore, we have

$$\underline{\varepsilon} = \int_0^{+\infty} dk \,(\text{Full Theory Prediction})\, e^{-\frac{k}{\Lambda}}$$

$$+ \int_0^{+\infty} dk \,(\text{Full Theory Prediction})\,(1 - e^{-\frac{k}{\Lambda}})$$

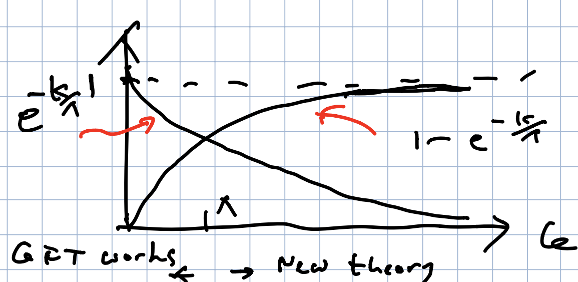
$$= \underbrace{4\pi \int_0^{+\infty} dk\, k^3 e^{-\frac{k}{\Lambda}}}_{\text{below } \Lambda \text{ we can use QFT}} + \int_0^{+\infty} dk \,[\text{Full Theory Prediction}]\,(1-e^{-\frac{k}{\Lambda}})$$

$$\equiv \underline{\varepsilon_B} + \underbrace{4\pi \int_0^{\infty} dk\, k^3 e^{-\frac{k}{\Lambda}}}$$

what experimentally can be measured (Universe expansion) & independent of our scale choice

something we don't know but we parametrize as a Bare energy density

what QFT can "predict" depends on an artificial scale choice Λ



$e^{-\frac{k}{\Lambda}}$ acts as a UV cut off to regulate the UV divergence.

$$\Rightarrow \quad \mathcal{E} = \mathcal{E}_B + 4\pi \cdot \Lambda^4 \quad \text{in QFT}$$

$$\mathcal{E}_B = -4\pi\Lambda^4 + \mathcal{E}$$

- Λ choice here is quite arbitrary and artificial. It is a scale at which we estimate the break-down of QFT.

- Therefore ANY physical observables can NOT depend on Λ (Here $\mathcal{E}$ is Λ-independent)

- $\mathcal{E}_B$ is NOT physically observable. and $\mathcal{E}_B \to \infty$ as $\Lambda \to \infty$ to cancel the divergence from k-integral

- by measuring $\mathcal{E}$ we can fix $\mathcal{E}_B$. but QFT can NOT predict $\mathcal{E}$!

We can put in $\mathcal{E}_B$ at the very beginning in the Lagrangian Density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$\to \quad \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \mathcal{E}_B$$

Hence

$$\mathcal{H} \to \mathcal{H} + \mathcal{E}_B$$

Renormalization procedure, a procedure to reduce UV-dependence

- introduce a UV regulator to turn divergence to "poles". e.g. in Vac Energy : $\infty \to \Lambda^4$ poles

- introduce the bare parameter to absorb the "poles". since the bare parameter is not observable. it's always justified to do so. Fix the finite piece of the parameter by compare with the experimental measured value.

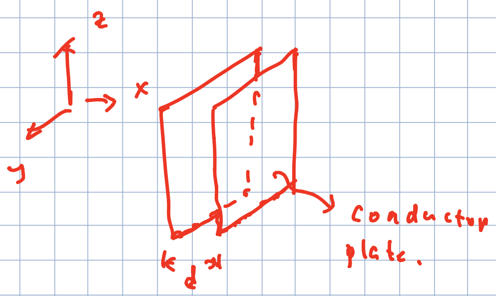
- Those parameters are NOT predictable in QFT

- But once we fix the parameters, we can make predictions using Q.F.T. And the predictions are UV-insensitive.

* If in a theory, by introducing a finit set of parameters, we can absorb all infinities. This theory is called renormalizable.

* It is totally fine to have an effective theory non-renormalizable. in practice.

2. Casimir effects



$A^i(0) = A^i(d) = 0$

$\Rightarrow k_x = \frac{\pi n_x}{d}$

$\int dk_x \Rightarrow \frac{\pi}{d}\sum_{n_x}$

$\Lambda \rightarrow$ UV regulator

$d \rightarrow$ IR regulator

The energy between the plate is

$$E(d) = \frac{1}{2}\int d^3k \, |\vec{k}| \, e^{-|\vec{k}|/\Lambda} + \rho_{Bare}(\Lambda) d$$

$$= \sum_{n_x}\int dk_y \, dk_z \sqrt{\left(\frac{n_x\pi}{d}\right)^2 + k_y^2 + k_z^2} \; e^{-\frac{\sqrt{\left(\frac{n_x\pi}{d}\right)^2 + k_y^2 + k_z^2}}{\Lambda}} +$$

let's do a one-dimension analogous.

$$\varepsilon_{vac} \equiv \frac{E}{L} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{2\pi} |k| + \rho^{Bare}$$

- Introduce a regulator : $e^{-\frac{k}{\Lambda}} = e^{-ak}$ $a = \frac{1}{\Lambda}$

$$\varepsilon_{vac} = \frac{E}{L} = \int_0^{+\infty} \frac{dk}{2\pi} k e^{-ak} + \rho^{Bare}$$

$$= \frac{\Lambda^2}{2\pi} + \rho^{Bare}$$

$\delta\rho \xrightarrow{\Lambda\to\infty} \infty$ ρ_R infinite

- determine the Bare parameter.

$$\rho^{Bare} = -\frac{\Lambda^2}{2\pi} + \varepsilon_{vac} = \delta\rho + \rho_R$$

– Now we calculate the energy between two conductor plate.

$$\boxed{k = \frac{\pi n}{d}}$$

$$\frac{E(d)}{d} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{2\pi} |k| e^{-\frac{k}{\lambda}} + \rho^{Bare}$$

$$= \frac{1}{2\pi}\frac{\pi}{d}\sum_n^\infty \frac{\pi n}{d} e^{-\frac{a\pi n}{d}} + \rho^{Bare}$$

$$= \frac{1}{2d}\sum_n^\infty -\partial_a e^{-\frac{a\pi n}{d}} + \rho^{Bare}$$

$$= \frac{1}{2d}(-\partial_a)\sum_n^\infty e^{-\frac{a\pi n}{d}} + \rho^{Bare}$$

$$= \left(\frac{1}{2d}\right)(-\partial_a)\frac{1}{1-e^{-\frac{a\pi}{d}}} + \rho^{Bare}$$

$$\frac{E(d)}{d} = \left(\frac{1}{2d}\right)\frac{\pi}{d}\frac{1}{(1-e^{-\frac{a\pi}{d}})^2}e^{-\frac{a\pi}{d}} + \rho^{Bare}$$

$$\Rightarrow \frac{E(d)}{d} = \left(\frac{1}{2d}\right)\left(\frac{d}{\pi a^2} - \frac{\pi}{12d}\right) + \rho^{Bare} + \mathcal{O}\left(\frac{1}{\Lambda}\right)$$

$$= \frac{1}{d}\left[\frac{d\Lambda^2}{2\pi} - \frac{\pi}{24d}\right] + \rho^{Bare} + \mathcal{O}\left(\frac{1}{\Lambda}\right)$$

$$= \frac{\Lambda^2}{2\pi} - \frac{\pi}{24d^2} + \left(-\frac{\Lambda^2}{2\pi}\right) + \varepsilon_{vac} + \mathcal{O}\left(\frac{1}{\Lambda}\right)$$

$$\frac{E(d)}{d} = -\frac{\pi}{24d^2} + \varepsilon_{vac} + \mathcal{O}\left(\frac{1}{\Lambda}\right)$$

→ insensitive to UV !!

Now we can take $\Lambda \to \infty$ safely

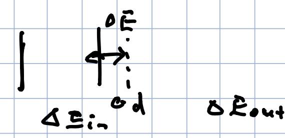
$$\frac{E(d)}{d} = -\frac{\pi}{24d^2} + \varepsilon_{vac}$$

→ that's QFT prediction!

Well it is hard to measure.

However we can measure the force by changing the distance d.

$$F = -\frac{\delta E}{\delta d}$$



Here

$$\delta E = \delta E_{in} + \delta E_{out}$$

$$\delta E_{out} = \mathcal{E}_{out}(-\delta d) = -\mathcal{E}_{vac}\,\delta d$$

$$\delta E_{in} = -\frac{\pi}{24(d+\delta d)} + \frac{\pi}{24\,d} + \mathcal{E}_{vac}(d+\delta d) - \mathcal{E}_{vac}\cdot d$$

$$= -\frac{\pi}{24}\left[\frac{1}{d}\right]\cdot\left[\frac{1}{1+\frac{\delta d}{d}} - 1\right] + \mathcal{E}_{vac}\,\delta d$$

$$= -\frac{\pi}{24}\cdot\left[\frac{1}{d}\right]\cdot\left[-\frac{\delta d}{d}\right] + \mathcal{E}_{vac}\cdot\delta d + \mathcal{O}\left(\frac{\delta d^2}{d^2}\right)$$

$$\delta E = \delta E_{in} + \delta E_{out} = \frac{\pi}{24}\,\frac{\delta d}{d^2} + \mathcal{O}\left(\frac{\delta d^2}{d^2}\right)$$

$$F = \lim_{\delta d\to 0} -\frac{\delta E}{\delta d} = -\frac{\pi}{24}\,\frac{1}{d^2}$$

$\rightarrow$ UV ignorant. Another QFT prediction!

- Casimir force due to vacuum fluctuations!
- independent of the regulator we choose
  it is a physical observable
  See Schwartz QFT Chapter 15 for more discussion.

— Regulator Independence Check:

- Introduce a regulator : $R(\frac{k}{\Lambda})$ demand $k \to \infty \quad kR^{(n)} \to 0$
$k \to 0 \quad R \to 1$

R only suppresses UV. Do NOT CHANGE IR physics

$$\varepsilon_{vac} \equiv \frac{E}{L} = \int_0^{+\infty} \frac{dk}{2\pi}\, k\, R\left(\frac{k}{\Lambda}\right) + \rho_R^{Bare}$$

- determin the Bare parameter.

$$\varepsilon_{vac} = \frac{E}{L} = \Lambda^2 \int_0^{+\infty} \frac{dk}{2\pi\Lambda} \frac{k}{\Lambda} R\left(\frac{k}{\Lambda}\right) + \rho^{Bare}$$

$$= \frac{\Lambda^2}{2\pi} \int_0^{+\infty} \frac{d}{\Lambda}\, dn\ \frac{dn}{\Lambda} R\left(\frac{dn}{\Lambda}\right) \leftarrow$$ we let $k = \frac{n\pi}{d}$
here n is continuous.

$$\Rightarrow \rho^{Bare} = -\frac{\Lambda^2}{2\pi} \cdot F(R, \Lambda) + \varepsilon_{vac}$$

- Now predict $E(d)$

$$\frac{E(d)}{d} = \frac{\Lambda^2}{2\pi} \int_0^{+\infty} \frac{dk}{\Lambda} \frac{k}{\Lambda} R\left(\frac{k}{\Lambda}\right) + \rho^{Bare}$$

$$\Rightarrow \frac{\Lambda^2}{2\pi} \frac{\pi}{d} \cdot \sum_n \frac{1}{\Lambda} \frac{n\pi}{d\Lambda} R\left(\frac{n\pi}{d\Lambda}\right) + \rho^{Bare}$$

Euler-Maclaurin
$$= \frac{\Lambda^2}{2\pi} \lim_{N\to\infty} \left(\frac{\pi}{d}\frac{1}{\Lambda} \sum_{n=1}^{N} \frac{n\pi}{d\Lambda} R\left(\frac{n\pi}{d\Lambda}\right) - \frac{\pi}{d}\frac{1}{\Lambda} \int_0^N dn\, \frac{n\pi}{d\Lambda} R\left(\frac{n\pi}{d\Lambda}\right) \right) + \varepsilon_{vac}$$

$$\sum_n^N f(n) - \int_0^N dx\, f(x) = \frac{f(0)+f(N)}{2} + \frac{f'(N) - f'(0)}{12} + \dots$$

$$= \frac{\Lambda^2}{2\pi} \frac{\pi}{d} \frac{1}{\Lambda} \left(\frac{F(0)+F(N)}{2} + \frac{F'(N) - F'(0)}{12} + \dots \right) + \varepsilon_{vac}$$

with $F(n) = \frac{n\pi}{d\lambda} R\left(\frac{n\pi}{d\lambda}\right)$

$$\begin{cases} F(0) = 0\,, \quad F(N) = \dfrac{N\pi}{d\lambda} R\left(\dfrac{N\pi}{d\lambda}\right) \xrightarrow{N\to\infty} 0 \\ F'(N) = \dfrac{\pi}{d\lambda} R\left(\dfrac{N\pi}{d\lambda}\right) + \dfrac{N\pi}{d\lambda}\dfrac{\pi}{d\lambda} R'\left(\dfrac{N\pi}{d\lambda}\right) \xrightarrow{N\to\infty} 0 \\ F'(0) = \dfrac{\pi}{d\lambda} R(0) + 0 \rightarrow \dfrac{\pi}{d\lambda}\, 1 \end{cases}$$

$$\Rightarrow \quad \frac{E(d)}{\lambda} = \frac{\cancel{\Lambda}}{2\pi}\,\frac{\pi}{d}\,\frac{1}{\cancel{\lambda}}\left(-\frac{\pi}{d\cancel{\lambda}}\cdot\frac{1}{12}\right) + \varepsilon_{vac}$$

$$\frac{E(d)}{d} = -\frac{\pi}{24}\,\frac{1}{d^2} + \varepsilon_{vac}$$

independent of the explicit form of R

thus independent of the regulator !