Gauge Redundancy / Gauge Symmetry

$\mathcal{L}$ is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$

depends on x
NOT a const. (or global)
But Local.
usually require
$\lambda(x) \to 0$, $x \to \infty$

Since.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\longrightarrow \partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu} \lambda - \partial_\nu A_\mu - \cancel{\partial_\nu \partial_\mu} \lambda$$

$$= F_{\mu\nu} \quad \text{unchanged.}$$

and hence

$\mathcal{L}$ unchanged

More precisely, this is called Abelian $U(1)$ Gauge.

A_μ is called $U(1)$ gauge field.

Some other Gauge theories:

1. General Relativity
2. Yang-Mills theory. (Standard Model and etc.)

Gauge theories are fundamental building blocks of modern physics!

Comment 1 : Gauge symmetry forbids the mass of the Gauge field, since $mA^\mu A_\mu \to m(A^\mu + \partial^\mu\lambda)(A_\mu + \partial_\mu\lambda)$ is not invariant under the gauge transformation. Therefore $m A_\mu A^\mu$ is not allowed, if we demand the Gauge Symmetry to be exact.

need new ideas for the masses
→ spontaneous symmetry breaking!
(Higgs mechanism)

Comment 2 : Gauge symmetry will constrain the form of interactions between the Gauge field and the "matters", NOT all interactions are allowed. HW ?

Comment 3 : Gauge Symmetry is NOT a Symmetry! It's redundancy.

$$\partial^2 A_\nu - \partial_\nu \partial\cdot A = 0$$ does not fix A_μ.

if A_μ is a solution, $A_\mu + \partial_\mu\lambda$ are solutions

since $\partial^2 \partial_\nu\lambda - \partial_\nu \partial^2\lambda = 0$

So we have infinite solutions connected by Gauge transformations. They corresponds to the same physical state. To make predictions, we have to fix a gauge !!

→ unlike "real" symmetries which are related to the transformations from one physical state to another! e.g. rotation, displacement . .

No Conservation law is related to Gauge Invariance!

Choices of Gauge (purely mathematical procedure, doesn't depend on free or interacting fields)

a. Coulomb Gauge: $\vec{\nabla}\cdot\vec{A} = 0$ (we can always choose a $\lambda(x)$ to make A^μ satisfy $\vec{\nabla}\cdot\vec{A}=0$)

proof of validity: if we have A'_μ satisfy, $\vec{\partial}\cdot\vec{A}' = f(x)$

Then we choose $A^\mu = A'^\mu + \partial^\mu\lambda$

with $\vec{\nabla}^2\lambda = -f(x)$

This equation always has a unique solution.

If we require the boundary condition ($\lambda\to 0$, as $x\to\infty$)

Then. $\vec{\partial}\cdot A = \vec{\nabla}\cdot\vec{A}' + \vec{\partial}^2\lambda = f - f = 0$

Coulomb gauge entirely remove the Gauge redundancy.

Now we look at the E.O.M.

$$\partial^2 A_\nu - \partial_\nu\partial^\mu A_\mu = 0$$

$$\Rightarrow \begin{cases} \nu=0: & \partial^2 A_0 - \partial_0(\partial_0 A_0 + \cancelto{0}{\vec{\partial}\cdot\vec{A}}) = 0 \\ \nu=i: & \partial^2 A_i + \partial_i(\partial_0 A_0 + \cancelto{0}{\vec{\partial}\cdot\vec{A}}) = 0 \end{cases}$$

$$\Rightarrow \quad \cancel{\partial_0^2 A_0} - \vec{\partial}^2 A_0 - \cancel{\partial_0^2 A_0} = 0 \quad\Rightarrow\quad \vec{\partial}^2 A_0 = 0 \Rightarrow A_0 = 0$$

$$\partial_0^2\vec{A} - \vec{\partial}^2\vec{A} + \partial_0\vec{\partial}A_0 = 0 \Rightarrow (\partial_0^2 - \vec{\partial}^2)\vec{A} = 0$$

↳ similar to the massless K.G. eq.

Originally we have A_μ, $\mu = 0,1,2,3$. 4-degrees of freedom

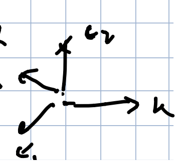
but $A_0 = 0$ is not dynamical.

and $\vec{\partial}\cdot\vec{A} = 0$ to remove another degrees of freedom

$\Rightarrow \vec{k}\cdot\vec{A} = 0$

left with 2-degrees of freedom. consistent with

2-polarizations of E&M fields!



(we can always choose a $\lambda(x)$ to make A^μ satisfy $\partial^\mu A_\mu = 0$)

b. Lorentz Gauge: $\partial^\mu A_\mu = 0$

proof of validity: if we have A'_μ satisfy $\partial^\mu A'_\mu = f(x)$

Then we choose $A^\mu = A'^\mu + \partial^\mu \lambda$

with $\partial^\mu \partial_\mu \lambda = -f(x)$

This equation always has solutions.

but not unique! Since we can always add λ_1 satisfies $\partial^\mu \partial_\mu \lambda_1 = 0$

Then. $\partial^\mu \cdot A_\mu = \partial^\mu \cdot A'_\mu + \partial^2 \lambda = f - f = 0$

$$\partial^2 A_\nu - \partial_\nu \cancel{\partial^\mu A_\mu}^{\,0} = 0$$

$\Rightarrow \quad \partial^2 A_\mu = 0 \quad \rightarrow$ make Lorentz structure obvious

similar to massless K.G. eqn.

but degrees of freedom ???

need to impose additional constraints like $A_0 = 0$

There are other possible choices of gauge we do not list here.

Hamiltonian Dynamics

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$\begin{cases} \pi^0 = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 & \text{Since No } \dot{A}_0 \text{ in } F_{\mu\nu}. \\ \pi^i = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i, & \text{note } F^{0i} = -F_{0i} \end{cases}$$

$i = 1, 2, 3$

Again. A_0 is NOT Dynamical, Since $\pi^0 = 0$

$$H = \int d^3x \; \pi^i \dot{A}_i - \mathcal{L}$$

$$= \int d^3x \; E^i \dot{A}_i - \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$= \int d^3x \; \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 - \vec{E}\cdot\vec{E} - \vec{E}\cdot\dot{\vec{A}}$$

$$= \int d^3x \; \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 - \vec{E}\cdot\underbrace{(-\dot{\vec{A}} - \vec{\nabla}A_0)}_{=\vec{E}} - \vec{E}\cdot\dot{\vec{A}}$$

$$= \int d^3x \; \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 + \vec{E}\cdot\vec{\nabla}A_0$$

$$= \int d^3x \; \frac{1}{2}(\vec{E}^2 + \vec{B}^2) - \underline{A_0}\,(\underline{\vec{\nabla}\cdot\vec{E}})$$

$\vec{\nabla}\cdot\vec{\pi} = 0$, constraints

Lagrangian multiplier

$$= \int d^3x \; \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

$$= \int d^3x \; \frac{1}{2}\vec{\pi}^2 + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2$$ Positive definite!

- Quantization of free E & M field

Try to quantize E × M:

$\mathcal{H}(\vec{\pi}, \vec{A})$, fix $\vec{\nabla}\cdot\vec{A}=0$

$$[\underbrace{A_i}_{\text{component}}(\underbrace{x}_{\text{label}}), \pi^j(y)] \stackrel{?}{=} i\delta^i_j\, \delta(\vec{x}-\vec{y})$$

$\Leftarrow$ a wild guess

Quantum Mechanics :

$H[p_i, q_i]$

$[q_i, p_i] = i\delta_{ij}$

Canonical commutation

How we do Quantization

Consistency check :

$$[A_i(x), \pi_j(y)] \stackrel{?}{=} -i\,\delta_{ij}\,\delta(\vec{x}-\vec{y})$$

$$\partial_i\partial_j [A_i(x), \pi_j(y)] \stackrel{?}{=} -i\,\partial_i\partial_j\delta_{ij}\,\delta(\vec{x}-\vec{y})$$

$$[\vec{\nabla}\cdot\vec{A}, \vec{\nabla}\cdot\vec{\pi}] \stackrel{?}{=} -i\,\vec{\nabla}^2\,\delta(\vec{x}-\vec{y})$$

$[\vec{\nabla}\cdot\vec{A}, \vec{\nabla}\cdot\vec{\pi}] = 0$ due to Coulomb gauge

$\vec{\nabla}^2\delta(\vec{x}-\vec{y}) \neq 0$!! can not be correct.

inconsistent with our gauge choice

proposing the naive canonical commutation relation to quantize the E & M field in Coulomb Gauge, seems NOT self-consistent.

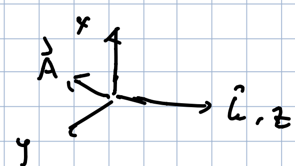
What's the problem?

$$[A_i(x), \pi_j(y)] \overset{?}{=} -i\,\delta_{ij}\,\delta(\vec{x}-\vec{x})$$

try to quantize 3- components.

but we only have 2- degrees of freedom !!

let's set the z-component to be along with the propagation direction of the EM wave



The $\vec{A}$ will be lying in the x-y plane only.

Therefore, a more reasonable guess will be

$$[A_i(x), \pi_j(y)] = -i\,\delta^{Tr}_{ij}\,\delta(\vec{x}-\vec{y}), \quad i=1,2.$$

$$\delta^{Tr}_{ij} = \begin{bmatrix}1&&\\&1&\\&&0\end{bmatrix} = \begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} - \begin{bmatrix}0&&\\&0&\\&&1\end{bmatrix}$$

$$= \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}$$ in a rotational invariant form.

propose $\Rightarrow$

$$[A_i(x), \pi_j(y)] = i\left(\delta_{ij} - \frac{\partial_i \partial_j}{\vec{\partial}^2}\right)\delta(\vec{x}-\vec{y})$$

Consistency: $\partial_i [A_i(x), \pi_j(y)] = i(\partial_j - \partial_j)\,\delta(\vec{x}-\vec{y}) = 0$ ✓

Same for $\partial_j [A_i(x), \pi_j(y)] = 0$ ✓

$$[A_i(x), \pi_j(y)] = -i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\vec{\partial}^2}\right)\delta(\vec{x} - \vec{y})$$

$+ \quad \mathcal{H}(\vec{\pi}, \vec{A}) = \int d^3x \, \frac{1}{2}\left[\vec{\pi}^2 + (\vec{\nabla}\times\vec{A})^2\right]$

$=$ Q.F.T. for free E .

Now let's consider expanding A_i in terms of plane waves:

$$\vec{A}_i = \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}}\left(\vec{\epsilon}_r^{\,i}\, a_r(k)\, e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t} + \vec{\epsilon}_r^{\,*i}\, a_r^*(k)\, e^{-i\vec{k}\cdot\vec{x}} e^{i\omega t}\right)$$

$$\vec{E}_i = i\int d^3\vec{k} \sqrt{\frac{\omega_k}{2}}\left(\vec{\epsilon}_r^{\,i}\, a_r(k)\, e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t} - \vec{\epsilon}_r^{\,*i}\, a_r^*(k)\, e^{-i\vec{k}\cdot\vec{x}} e^{i\omega t}\right)$$

Here $\omega_k = |\vec{k}|$, $r = 1, 2$ it's easy to check the $\partial^2 A_i = 0$.

And the Gauge condition $\vec{\nabla}\cdot\vec{A}$ requires $\vec{k}\cdot\vec{\epsilon}_r = 0$

We note that, by proposing

$[a_r, a_{r'}] = 0 = [a_r^*, a_{r'}^*]$

$[a_r(k), a_{r'}^*(k')] = \delta_{rr'}\,\delta^3(\vec{k} - \vec{k}')$

Nothing but the creation - anhilation operators in the harmonic oscillator!

We can satisfy

$$[A_i(x), E_j(y)] = -i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta(\vec{x} - \vec{y})$$

proof.

$$[A_i, E_j] = i\int \frac{d^3k}{\sqrt{2\omega_k}}\, d^3k' \sqrt{\frac{\omega_{k'}}{2}}\;\Big[\vec\epsilon_i^{\,r} a_r(k) e^{i\vec k\cdot\vec x} + \vec\epsilon_i^{\,*r} a_r^*(k) e^{-i\vec k\cdot\vec x},\ \vec\epsilon_j^{\,r'} a_{r'}(k') e^{i\vec k'\cdot\vec y} - \vec\epsilon_j^{\,*r'} a_{r'}^*(k') e^{-i\vec k'\cdot\vec y}\Big]$$

$$= i\int \frac{d^3k}{\sqrt{2\omega_k}}\, d^3k' \sqrt{\frac{\omega_{k'}}{2}}\;\Big\{ \vec\epsilon_i^{\,r}\vec\epsilon_j^{\,r'} \cancel{[a_r(k), a_{r'}(k')]}^{\,0}\, e^{i\vec k\cdot\vec x} e^{i\vec k'\cdot\vec y}$$

$$- \vec\epsilon_i^{\,r}\vec\epsilon_j^{\,*r'} [a_r(k), a_{r'}^*(k')]\, e^{i\vec k\cdot\vec x} e^{-i\vec k'\cdot\vec y}$$

$$- \vec\epsilon_i^{\,*r}\vec\epsilon_j^{\,r'} [a_{r'}(k'), a_r^*(k)]\, e^{-i\vec k\cdot\vec x} e^{i\vec k'\cdot\vec y}$$

$$- \vec\epsilon_i^{\,*r}\vec\epsilon_j^{\,*r'} \cancel{[a_r^*(k), a_{r'}^*(k')]}^{\,0}\, e^{-i\vec k\cdot\vec x} e^{-i\vec k'\cdot\vec y} \Big\}$$

$$= i\int \frac{d^3k}{\cancel{\sqrt{2\omega_k}}}\, \cancel{d^3k'} \sqrt{\frac{\omega_k}{2}}\;\Big(- \vec\epsilon_i^{\,r}\vec\epsilon_j^{\,*r'} \delta_{rr'}\, \delta^3(\vec k-\vec k')\, e^{i\vec k(\vec x-\vec y)} - \vec\epsilon_i^{\,*r}\vec\epsilon_j^{\,r'} \delta_{rr'}\, \delta^3(\vec k-\vec k')\, e^{-i\vec k(\vec x-\vec y)} \Big)$$

$$= -\frac{i}{2}\int d^3k \Big(\underline{\vec\epsilon_i^{\,r}\vec\epsilon_j^{\,*r'}\delta_{rr'}}\, e^{i\vec k(\vec x-\vec y)} + \underline{\vec\epsilon_i^{\,*r}\vec\epsilon_j^{\,r'}\delta_{rr'}}\, e^{-i\vec k(\vec x-\vec y)} \Big)$$

$$\sum \vec\epsilon_i \vec\epsilon_j^{\,r'*} \delta_{rr'} = \left(\delta_{ij} - \frac{k_i k_j}{|\vec k|^2}\right)$$

$$= -\frac{i}{2}\int d^3k \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\left(e^{i\vec k(\vec x-\vec y)} + e^{-i\vec k(\vec x-\vec y)}\right)$$

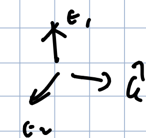
$$= -\frac{i}{2}\int d^3k \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}^{**}\right)\left(e^{i\vec k(\vec x-\vec y)} + e^{-i\vec k(\vec x-\vec y)}\right)$$

$$= -i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta(\vec x-\vec y)$$

$(\partial^2)^{-1}$ defined as $(\partial^2)^{-1}(\partial^2) = 1$

* To see this, we can choose a frame in which

$$\vec{\epsilon}_1 = (1\;0\;0)\,,\quad \vec{\epsilon}_2 = (0,1,0)\,,\quad \frac{\vec{k}}{|\vec{k}|} = (0,0,1)$$



Therefore $\vec{k}\cdot\vec{\epsilon}_r = 0$ is satisfied.

$$\epsilon^r_i \epsilon^{r*}_j = \epsilon^1_i \epsilon^{1*}_j + \epsilon^2_i \epsilon^{2*}_j$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\;0\;0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\;1\;0)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}$$

** To think of $(\vec{\nabla}^2)^{-1}$, we consider

$$\vec{\nabla}^2 \Phi = \rho \quad \overset{\text{by definition}}{\Longrightarrow} \quad \Phi = \frac{1}{\nabla^2}\rho \qquad - ①$$

Let $\Phi = \int d^3\vec{k}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\Phi}(\vec{k})$, $\rho = \int d^3\vec{k}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\rho}(\vec{k})$ $\quad - ②$

$$\Rightarrow \vec{\nabla}^2 \Phi = \int d^3\vec{k}\, (-\vec{k}^2)\, \tilde{\Phi}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} = \int d^3\vec{k}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\rho}(\vec{k})$$

$$\Rightarrow (-\vec{k}^2)\tilde{\Phi} = \tilde{\rho} \Rightarrow \tilde{\Phi} = \frac{1}{-\vec{k}^2}\tilde{\rho} \qquad - ③$$

$$\Rightarrow \frac{1}{\vec{\nabla}^2} \xrightarrow{F.T.} -\frac{1}{k^2}$$

or
$$\int d^3\vec{k}\, \underbrace{\frac{1}{\vec{\nabla}^2}}\, \tilde{\rho}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} = \int d^3\vec{k}\, \underbrace{\frac{1}{-k^2}}\, \tilde{\rho}(k)\, e^{i\vec{k}\cdot\vec{x}}$$

From ①, ② and ③

Now we consider the Hamiltonian in terms of a, a^*,

Free

$\int \vec{\pi}^2 d^3\vec{x}$

$$= \int d^3\vec{x} \quad i\int d^3\vec{k} \sqrt{\frac{\omega_k}{2}} \left(\vec{\epsilon}_i^{\,r} a_r(k) e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t} - \vec{\epsilon}_i^{\,*r} a_r^*(k) e^{-i\vec{k}\cdot\vec{x}} e^{i\omega t} \right)$$

$$i\int d^3k' \sqrt{\frac{\omega_{k'}}{2}} \left(\vec{\epsilon}_i^{\,r'} a_{r'}(k') e^{i\vec{k}'\cdot\vec{x}} e^{-i\omega' t} - \vec{\epsilon}_i^{\,*r'} a_{r'}^*(k') e^{-i\vec{k}'\cdot\vec{x}} e^{i\omega' t} \right)$$

$$= -\frac{1}{2}\int d^3\vec{x} \int d^3\vec{k}\, d^3\vec{k}' \sqrt{\omega_k \omega_{k'}} \Big(\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'} a_r a_{r'} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} e^{-i(\omega+\omega')t}$$

$$+ \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,*r'} a_r^* a_{r'}^* e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}} e^{i(\omega+\omega')t}$$

$$- \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'} a_r^* a_{r'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} e^{i(\omega-\omega')\cdot t}$$

$$- \vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'} a_r a_{r'}^* e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} e^{-i(\omega-\omega')\cdot t} \Big)$$

Here $a_r \equiv a_r(\vec{k})$, $a_{-r} \equiv a_r(-\vec{k})$

use $\int d^3k \int d^3x \; e^{ikx} = \int d^3k\, \delta^3(k) = 1$

$$= -\frac{1}{2} \int d^3\vec{k}\, \omega_k \Big(\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'} a_r a_{-r'} e^{-2i\omega t}$$

$$+ \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,*r'} a_r^* a_{-r'}^* e^{2i\omega t}$$

$$- \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'} a_r^* a_{r'} - \vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'} a_r a_{r'}^* \Big)$$

$$\int d^3\vec{x}\, (\vec{\partial}\times\vec{A})^2 = \int d^3\vec{x} \quad \epsilon_{ijk}\,(\partial_j A_k)\;\epsilon_{ilm}\,(\partial_l A_m)$$

$$= \int d^3\vec{x} \quad (\partial_j A_k)(\partial_j A_k) - (\partial_j A_k)(\partial_k A_j)$$

$$= \int d^3\vec{x} \quad -\vec{A}\,\vec{\partial}^2\vec{A} \quad + A_k\,\partial_k\,(\cancel{\vec{\nabla}\cdot\vec{A}})^{0} \quad \text{Coulumb Gauge}$$

$$= \int d^3\vec{x} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(\vec{\epsilon}^{\,r}_i\, a_r(k)\, e^{i\vec{k}\cdot\vec{x}}\, e^{-i\omega t} + \vec{\epsilon}^{\,*r}_i\, a^*_r(k)\, e^{-i\vec{k}\cdot\vec{x}}\, e^{i\omega t} \right)$$

$$\int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}}\, \omega_{k'}^2 \left(\vec{\epsilon}^{\,r'}_i\, a_{r'}(k')\, e^{i\vec{k}'\cdot\vec{x}}\, e^{-i\omega' t} + \vec{\epsilon}^{\,*r'}_i\, a^*_{r'}(k')\, e^{-i\vec{k}'\cdot\vec{x}}\, e^{i\omega' t} \right)$$

$$= \int d^3\vec{x} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}}\, \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}}\, \omega_{k'}^2 \Big(\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'}\, a_r a_{r'}\, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\, e^{-i(\omega+\omega')t}$$

$$+ \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,*r'}\, a^*_r a^*_{r'}\, e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}\, e^{i(\omega+\omega')t}$$

$$+ \vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'}\, a_r a^*_{r'}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\, e^{-i(\omega-\omega')t}$$

$$+ \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'}\, a^*_r a_{r'}\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}\, e^{i(\omega-\omega')\cdot t} \Big)$$

$$= \frac{1}{2}\int d^3\vec{k}\;\omega_k \Big(\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'}\, a_r a_{-r'}\, e^{-i2\omega t} + \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,*r'}\, a^*_r a^*_{-r'}\, e^{i2\omega t}$$

$$+ \vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'}\, a_r a^*_{r'} + \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'}\, a^*_r a_{r'} \Big)$$

Therefore

$$H_{free} = \int d^3\vec{x} \;\; \tfrac{1}{2}\vec{\pi}_T^2 \; + \; \tfrac{1}{2}(\vec{\nabla}\times\vec{A})^2$$

$$= \frac{1}{4}\int d^3\vec{k}\; \omega_k \Big(-\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'}\, \cancel{a_r a_{-r'}}\; e^{-2i\omega t}$$

$$- \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,*r'}\, \cancel{a_r^* a_{-r'}^*}\; e^{2i\omega t}$$

$$+ \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'}\, a_r^* a_{r'} \; + \; \vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'}\, a_r a_{r'}^* \Big)$$

$$+ \frac{1}{4}\int d^3\vec{k}\; \omega_k \Big(\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'}\, \cancel{a_r a_{-r'}}\; e^{-i2\omega t}$$

$$+ \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,*r'}\, \cancel{a_r^* a_{r'}^*}\; e^{i2\omega t}$$

$$+ \vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'}\, a_r a_{r'}^* \; + \; \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'}\, a_r^* a_{r'} \Big)$$

$$= \frac{1}{2}\int d^3\vec{k}\; \omega_k \Big[\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,*r'}\, a_r a_{r'}^* + \vec{\epsilon}^{\,*r}\cdot\vec{\epsilon}^{\,r'}\, a_r^* a_{r'} \Big]$$

$\vec{\epsilon}^{\,r}\cdot\vec{\epsilon}^{\,r'} = \delta^{rr'}$

$$= \frac{1}{2}\int d^3\vec{k}\; \omega_k \Big[a_r a_r^* + a_r^* a_r \Big]$$

$a_r a_r^* = a_r^* a_r + [a_r, a_r^*]$

$= a_r^* a_r + \delta_{rr}\,\delta(\vec{0})$

$$= \sum_{r=1}^{2}\int d^3\vec{k}\; \omega_k \Big[a_r^* a_r + \tfrac{1}{2}\delta_{rr}\,\delta(\vec{0}) \Big]$$

- independent of time
- infinite sum of Harmonics
- energy $= \infty$. $\Leftarrow \int d^3\vec{k}\; |\vec{k}|\; \delta(\vec{0})$. very first infinity in Q.F.T.

$\delta(\vec{0})$: IR divergence $x \to \infty$ $\quad \delta(\vec{0}) = \int d^3x\, e^{i\vec{x}\cdot\vec{p}}\big|_{\vec{p}=0} = \int d^3x = \text{Volume}$

$\int d^3\vec{k}\, |\vec{k}|$: UV divergence. $k \to \infty$

Fock Space:

by using the creation and the annihilation operators $a_r(\vec{k}), a_r^\dagger(\vec{k})$ which satisfy

$$[a_r, a_{r'}] = 0 = [a_r, a_{r'}^\dagger]$$

$$[a_r(k), a_{r'}^\dagger(k')] = \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')$$

and the vaccuum state $|0\rangle$ defined as

$$a_r(\vec{k})|0\rangle = 0$$

we can generate the fock space

$$\text{Fock Space} = \bigoplus_{n=0}^{\infty} \text{Hilbert Space for } n \text{ particles.}$$

For instance

$$a_r^\dagger(\vec{k})|0\rangle = \frac{1}{\sqrt{2\omega_k}}\,|1_r(\vec{k})\rangle$$

→ create a particle with momentum $\vec{k}$ and pol. r

(Hilbert space for 1 particle)

Here we have defined the normalization by

$$\langle 1_r(\vec{k}')|1_r(\vec{k})\rangle = 2\omega_k\, \delta^3(\vec{k}-\vec{k}')$$

and

$$\sum_r \int \frac{d^3\vec{k}}{2\omega_k}\, |1_r(\vec{k})\rangle\langle 1_r(\vec{k})| = 1$$

Let's check the consistency of the conventions we introduced

1. $\langle 0 | [a_r(k), a^\dagger_{r'}(k')] | 0\rangle = \langle 0 | a_r(k) a^\dagger_{r'}(k') | 0\rangle - \langle 0 | \cancel{a^\dagger_{r'}(k') a_r(k)} | 0\rangle$ $\nearrow 0$

$$= \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2\omega_{k'}}} \langle 1_r(k) | 1_{r'}(k')\rangle = \frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}} 2\omega_k \delta_{rr'} \delta^3(\vec{k}-\vec{k}')$$

$$= \delta_{rr'} \delta^3(\vec{k}-\vec{k}')$$

agree with $[a_r(k), a^\dagger_{r'}(k')] = \delta_{rr'} \delta^3(\vec{k}-\vec{k}')$

2. $$|1_r(\vec{k})\rangle = \sum_{r'} \int d^3\vec{p} \, \frac{1}{2\omega_p} |1_{r'}(\vec{p})\rangle \langle 1_{r'}(\vec{p}) | 1_r(\vec{k})\rangle$$

$$= \sum_{r'} \int d^3\vec{p} \, \cancel{\frac{1}{2\omega_p}} |1_{r'}(\vec{p})\rangle \cancel{2\omega_k} \delta_{rr'} \delta^3(\vec{k}-\vec{p})$$

$$= |1_r(\vec{k})\rangle$$ ✓

From fields to particles:

$$\langle 1_r(\vec{k}) | \vec{A}(x) |0\rangle \longrightarrow |\vec{x}\rangle \vec{\epsilon}^{\,*}_r \cdot e^{i\omega_k t}$$

$$= \sum_{r'} \langle 1_r(\vec{k}) | \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}} \vec{\epsilon}^{\,*r'} a^\dagger_{r'}(\vec{k}') e^{-i\vec{k}'\cdot\vec{x}} e^{i\omega_{k'} t} |0\rangle$$

$$= \sum_{r'} \langle 1_r(\vec{k}) | \int \frac{d^3 k'}{2\omega_{k'}} \vec{\epsilon}^{\,*r'} e^{-i\vec{k}'\cdot\vec{x}} e^{i\omega_{k'} t} | 1_{r'}(\vec{k}')\rangle$$

$$= \sum_{r'} \int \frac{d^3\vec{k}'}{2\omega_{k'}} \vec{\epsilon}^{\,*r'} e^{-i\vec{k}'\cdot\vec{x}} \cdot e^{i\omega_{k'} t} 2\omega_k \delta_{rr'} \delta^3(\vec{k}-\vec{k}')$$

$$= \vec{\epsilon}^{\,*r} e^{-i\vec{k}\cdot\vec{x}} e^{i\omega_k \cdot t}$$

$$= e^{i\omega_k \cdot t} \langle 1_r(\vec{k}) | \vec{x}\rangle \vec{\epsilon}^{\,*}_r$$

wave function for a free photon with polarization r of ith component.

time evolution

From fields the particles!

More examples:

2-particle Hilbert space

$$a_r^\dagger(\vec{k})\, a_{r'}^\dagger(\vec{k}')\,|0\rangle = \frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}}\,|\,1_r(\vec{k}),\, 1_{r'}(\vec{k}')\,\rangle$$

$$a_r^\dagger(\vec{k})\, a_{r'}^\dagger(\vec{k})\,|0\rangle = \frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_k}}\,|\,1_r(\vec{k}),\, 1_{r'}(\vec{k})\,\rangle$$

→ create 2-particle states with different $\vec{k}$ or pol.

$$\frac{1}{\sqrt{2!}}\, a_r^\dagger(\vec{k})\, a_r^\dagger(\vec{k})\,|0\rangle = \frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_k}}\,|\,2_r(k)\,\rangle$$

symmetric factor.

→ create 2 particle with identical $\vec{k}$ and pol.r

⋮

$$\frac{[a_r^\dagger(\vec{k})]^n}{\sqrt{n!}}\,|0\rangle = \left(\frac{1}{\sqrt{2\omega_k}}\right)^n\,|\,n_r(\vec{k})\,\rangle$$

→ create n particle with identical $\vec{k}$ and pol.r

Also one can check that

$$\hat{H}\,|\,1_r(\vec{k})\rangle = \sum_{r'}\int d^3\vec{k}'\; \omega_{k'}\, a_{r'}^\dagger a_{r'}\;|1_r(\vec{k})\rangle + \ldots$$

$$= \sum_{r'}\int d^3\vec{k}'\; \omega_{k'}\, a_{r'}^\dagger\, a_{r'}\, a_r^\dagger \sqrt{2\omega_k}\,|0\rangle + \ldots$$

$$= \sum_{r'}\int d^3\vec{k}'\; \omega_{k'}\, a_{r'}^\dagger\,[a_{r'}, a_r^\dagger]\sqrt{2\omega_k}\,|0\rangle + \ldots$$

$$= \sum_{r'}\int d^3\vec{k}'\; \omega_{k'}\, a_{r'}^\dagger\, \delta_{rr'}\,\delta^{(3)}(\vec{k}'-\vec{k})\sqrt{2\omega_k}\,|0\rangle$$

$$= \omega_k\,|\,\underline{1_r(\vec{k})}\rangle$$

↳ is a eigenstate of $\hat{H}$ with eigenvalue ω_k for a single photon.

From the commutation relation, we can derive the results for a acting on the Fock states:

1. $$a_r(\vec{k})\,\frac{1}{\sqrt{2\omega_{k'}}}\,|1_{r'}(\vec{k}')\rangle = a_r(\vec{k})\,a^\dagger_{r'}(\vec{k}')|0\rangle$$

$$= [a_r(\vec{k}),\,a^\dagger_{r'}(\vec{k}')]|0\rangle + \cancelto{0}{a^\dagger_{r'}(\vec{k}')a_r(\vec{k})|0\rangle}$$

$$= \delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')|0\rangle$$

$\longrightarrow$ annihilate a particle if $k=k'$, $r=r'$, otherwise get 0.

2. $$a_r(\vec{k})\frac{1}{\sqrt{2\omega_{k'}}\sqrt{2\omega_{k''}}}|1_{r'}(\vec{k}'),1_{r''}(\vec{k}'')\rangle = a_r(\vec{k})\,a^\dagger_{r'}(\vec{k}')\,a^\dagger_{r''}(\vec{k}'')|0\rangle$$

$$= [a_r, a_{r'}]\,a^\dagger_{r''}|0\rangle + a^\dagger_{r'}\,a_r\,a^\dagger_{r''}|0\rangle$$

$$= \delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\frac{1}{\sqrt{2\omega_{k''}}}|1_{r''}(\vec{k}'')\rangle + a^\dagger_{r'}[a_r, a^\dagger_{r''}]|0\rangle + \cancelto{0}{a^\dagger_{r'}a^\dagger_{r''}a_r|0\rangle}$$

$$= \delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\frac{1}{\sqrt{2\omega_{k''}}}|1_{r''}(\vec{k}'')\rangle + \delta_{rr''}\,\delta(\vec{k}-\vec{k}'')\frac{1}{\sqrt{2\omega_{k'}}}|1_{r'}(\vec{k}')\rangle$$

3. $$a_r(\vec{k})\,\frac{a^{\dagger n}_{r'}(\vec{k}')}{\sqrt{n!}}|0\rangle = [a_r(\vec{k}),\,a^{\dagger n}_{r'}(\vec{k}')]\frac{1}{\sqrt{n!}}|0\rangle$$

$$+ a^{\dagger n}_{r'}(\vec{k}')\,a_r(\vec{k})|0\rangle$$

$$= n\cdot a^{\dagger\, n-1}_{r'}(\vec{k}')\,\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\frac{1}{\sqrt{n!}}|0\rangle$$

$$= \sqrt{n}\;\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\left(\frac{1}{\sqrt{2\omega_k}}\right)^{n-1}|(n-1)_r(\vec{k}')\rangle$$

$$\Rightarrow\; a_r(\vec{k})\left(\frac{1}{\sqrt{2\omega_k}}\right)^{n'}|n_{r'}(\vec{k}')\rangle = \sqrt{n_r(\vec{k})}\;\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\left(\frac{1}{\sqrt{2\omega_k}}\right)^{n-1}|(n-1)_{r'}(\vec{k}')\rangle$$

Also $$a^\dagger_r(\vec{k})\left(\frac{1}{\sqrt{2\omega_k}}\right)^{n_{r'}}|n_{r'}(\vec{k}')\rangle = \sqrt{n_{r'}(\vec{k})+1}\;\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\left(\frac{1}{\sqrt{2\omega_k}}\right)^{n+1}|(n+1)_{r'}(\vec{k}')\rangle$$

4. $$a_r^\dagger a_r \left(\frac{1}{\sqrt{2\omega_{r'}}}\right)|n_{r'}\rangle = a_r^\dagger n_{r'} a_{r'}^{\dagger\, n-1}(\vec{k}')\,\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\frac{1}{\sqrt{n!}}|0\rangle$$

$$= \delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\; n_r \frac{a_{r'}^{\dagger n}}{\sqrt{n!}}|0\rangle$$

$$= \underline{\underline{n_{r'}}}\;\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}')\frac{1}{\sqrt{2\omega_{k'}}}|n_{r'}(\vec{k}')\rangle$$

↪ counts the # of particles with $k' = k$, $r = r'$

5. define

$$\hat{N}_r \equiv a_r^\dagger a_r \text{ be the \# operator!}$$

And $$H_{free} = \sum_r \int d^3\vec{k}\left[a_r^\dagger a_r + \frac{1}{2}\delta_{rr}\delta(0)\right]\omega_k$$

$$= \sum_r \int d^3\vec{k}\left[\hat{N}_r(\vec{k}) + \cdots\right]\omega_k$$

$$\Rightarrow H_{free}|n_{r'}(k')\rangle = \sum_r \int d^3\vec{k}\left(\hat{N}_r(\vec{k}) + \cdots\right)\omega_k |n_{r'}(\vec{k}')\rangle$$

$$= \sum_r \int d^3\vec{k}\left(n_{r'}(\vec{k}')\,\delta_{rr'}\,\delta^3(\vec{k}-\vec{k}') + \cdots\right)\omega_k |n_{r'}(\vec{k}')\rangle$$

$$= \Big(n_{r'}(\vec{k}')\,\omega_{k'} + \underbrace{\cdots}_{\infty}\Big)\;|n_{r'}(\vec{k}')\rangle$$

measure the Energy for n photons with momentum $\vec{k}'$ and pol. r' $+\ \infty$.

$$\sum_r \int d^3\vec{k}\;\omega_k\,\frac{1}{2}\delta_{rr}\,\delta(0) = \underbrace{2}_{\text{2-polarization}}\times\frac{1}{2}\int d^3\vec{k}\;\omega_k\,\delta(\vec{0})$$

$$H_{free} |0\rangle = \sum_r \int d^3\vec{k} \; \frac{1}{2} \delta_{rr} \; \delta(0) \; \omega_{\vec{k}} \; |0\rangle$$

$$= 2 \times \frac{1}{2} \int d^3\vec{k} \; \omega_{\vec{k}} \; \delta(0) \; |0\rangle$$

measures the vacuum energy.

The energy of the vacuum is not 0 but ∞!!