Summary of early attempts

- both K.G and Dirac try to find a relativistic eqn. for single particle wave function.

- KG for spin-0, predicts the Energy level for π-atom.

- Dirac for spin-½, predicts the Energy level for Hydrogen.

- KG has the negative energy and negative probability problem.

  the ψ function for KG. eqn. can not be interpretated as a probability wave function.

- Dirac solves the negative energy problem by imposing the Dirac sea, but abandon the single particle picture.
  Also the ψ function for Dirac eqn. can not be interpretated as a single particle wave fuction.

- Causality Violation.

$$p = \langle x | e^{-iH(t-t_0)} | x_0 \rangle \neq 0 \quad \text{always!}$$

probablity amplitude to find a particle at x at time $= t$
when initially the particle localized at x_0 at time $= t_0$

However relativity: $p = 0$ if $x - x_0 > t - t_0$, since $v \leq 1$

$$\propto \int d^3k \, \langle x | k \rangle \langle k | e^{-iH(t-t_0)} | x_0 \rangle \propto \int d^3k \; e^{ikx} \cdot e^{-i\sqrt{k^2+m^2}(t-t_0)} e^{-ikx_0}$$

- when combine special relativity with quantum theory, we have to

1. abandon the probability wave function interpretation.
2. deal with infinite degrees of freedom problem

• The Birth of QFT

- attempts to quantize photon.
- number of photons is not conserved

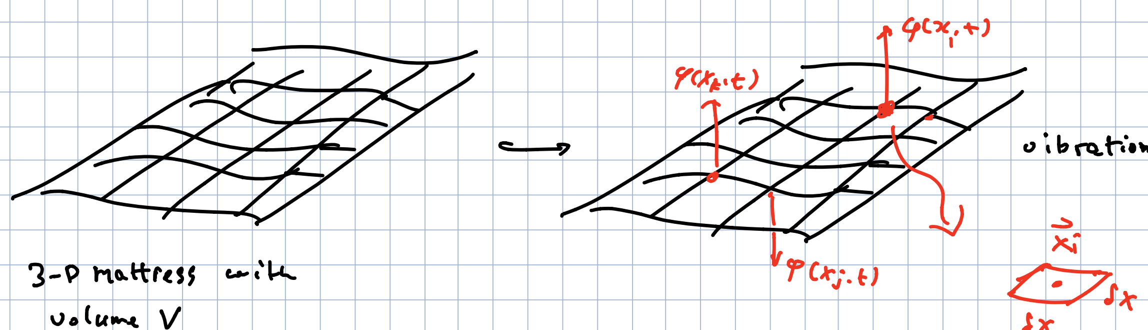
  e.g. $|\text{Atom}\rangle$ + energy $\to |\text{Atom}^*\rangle \longrightarrow |\text{Atom}\rangle + \gamma$

- known to be a field (E&M field) before,

  naturally we need a Quantum field theory.

• review of classic field theory (Local)

Let's study the "vibration of a 3-D mattress with volume V"



At the same time, we discretize it to N cells of size $\delta\vec{x}$, and indicate $\vec{x}_i$ the center of the cell. Now the problem reduces to a N-body system. $q_i(t) \equiv \varphi(t, \vec{x}_i)$, So the lagrangian for the system is given by

$$L = \sum_i^N L\left(q_i(t), \dot{q}_i(t), \underbrace{q_{i+1}(t)}_{\text{neibours sites}} \right).$$

Assume only nearest-neibour interation

I personally have no idea why we need to stick to this assumption.

$$q_{i+1}(t) = \varphi(\vec{x}_i, t) + \vec{\nabla}\varphi(\vec{x}_i, t)\cdot\delta\vec{x} + \mathcal{O}(\delta x)$$

ignoring higher orders in δx leads to linear spacial derivative only

$\nabla^2, \nabla^4 \ldots$ is possible! but will not be discussed in this course

$$L = \sum_i L\left[\varphi(x_i, t), \dot{\varphi}(x_i, t), \vec{\nabla}\varphi(\vec{x}_i, t) \right]$$

$$= \sum_i \delta^3\vec{x}\; \mathcal{L}\left(\varphi(x_i, t), \partial^\mu\varphi(x_i, t) \right)$$

$$\xrightarrow{N\to\infty,\ \delta\vec{x}\to 0} \int d^3x\; \mathcal{L}\left[\varphi(x, t), \partial^\mu\varphi(x, t) \right]$$

$\mathcal{L} = \frac{L}{\delta^3 x}$ density

Locality: Only depends on one x, but not $x_1, x_2, x_3 \ldots$

NOT neccessarily relativistic!! we can have non-relativistic-field theory. here we simply use the notation

field $\varphi(t, x)$: a function defined at every single point of space and time. x is only a label!

e.g. $A^\mu(x, t)$

$$S = \int dt\, L = \int dt \int d^3x\, \mathcal{L} = \int d^4x\, \mathcal{L}(\varphi, \partial^\mu\varphi)$$

We will also use the Hamiltonian (density) H or $\mathcal{H}$.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad\qquad H = p_i \dot{q}_i - L.$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \delta^3 x \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \longrightarrow d^3 x \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \equiv d^3 x \, \pi$$

$$H = p_i \dot{q}_i - L = \delta^3 x \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \delta^3 x \mathcal{L} . \rightarrow d^3 x \left(\pi \dot{\varphi} - \mathcal{L} \right) \equiv d^3 x \, \mathcal{H}.$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \qquad\qquad \underbrace{\mathcal{H}}_{\text{energy density}} = \pi \cdot \dot{\varphi} - \mathcal{L}.$$

We see the following dictionary from mechanics to the field theory:

	Classic Mechanics		Classic field theory
Lagrangian :	$q_i(t)$	$\longrightarrow$	$\varphi(x,t)$
	$L[q_i, \dot{q}_i]$	$\longrightarrow$	$\mathcal{L}[\varphi(x,t), \partial_\mu \varphi(x,t)]$
	t	$\longrightarrow$	x, t

Hamiltonian:

$$p_i(t) \longrightarrow \pi(x,t)$$

$$H(p_i, q_i) \longrightarrow \mathcal{H}[\pi, \varphi]$$

$$\{p_i, q_i\}_{\text{poisson}} \longrightarrow \{\pi, \varphi\}_{\text{poisson}}$$

Some examples:

example 1. Euler - Lagrangian equation

$$\frac{d}{dt}\frac{dL}{d\dot{q}_i} = \frac{dL}{dq_i} \longrightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

Prove:

Using the action principle $\delta S = 0$

$$\delta S = \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

$$= \int d^4x \, \mathcal{L}(\varphi + \delta\varphi \quad \partial_\mu \varphi + \partial_\mu \delta\varphi) - \mathcal{L}(\varphi, \partial_\mu \varphi)$$

$$= \int d^4x \, \cancel{\mathcal{L}(\varphi, \partial_\mu \varphi)} + \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\partial_\mu \delta\varphi - \cancel{\mathcal{L}(\varphi, \partial_\mu \varphi)}$$

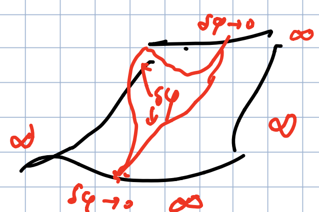
$$= \int d^4x \, \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi}\partial_\mu \delta\varphi$$

$$= \int d^4x \; \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \partial_\mu\left(\frac{\delta \mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right) - \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\delta\varphi$$

$$= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\varphi}\right]\delta\varphi + \underbrace{\partial_\mu\left(\frac{\delta\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right)}$$

arbitrary infinitesimal change vanishes as $x \to \infty$

total derivative vanishes as $x \to \infty$ since $\delta\varphi \to 0$.



Therefore

$$\delta S = 0 \quad \Rightarrow \quad \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi}.$$

example 2. conserved charge / current

$$Q = \frac{\partial L}{\partial \dot q}\delta q - F \quad \longrightarrow \quad j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi - F^\mu$$

if δq makes $\delta L = \frac{dF}{dt}$ — if $\delta\varphi$ makes $\delta\mathcal{L} = \partial_\mu F^\mu$

$\delta S = \int dt\, \delta L = \int dt \frac{dF}{dt} = 0$ — $\delta S = \int d^4x\, \delta\mathcal{L} = \int d^4x\, \partial_\mu F^\mu = 0$

proof

$$\delta\mathcal{L} = \partial_\mu F^\mu = \underbrace{\cancelto{0}{\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)}}_{\text{Euler-Lagrangian eqn.}}\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right)$$

$$\Rightarrow \partial_\mu F^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right)$$

$$\Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi - F^\mu\right) = 0$$

$$\Rightarrow \partial_\mu j^\mu = 0 \qquad \text{with} \qquad j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi - F^\mu$$

conserved current

example 3. free Klein-Gordon Field (complex)

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}, \qquad \varphi^* = \frac{\varphi_1 - i\varphi_2}{\sqrt{2}}$$

We choose φ and φ^* be independent.

equation of motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^*} = \partial_\mu \left[\frac{\partial}{\partial \partial_\mu \varphi} \left(\partial_\alpha \varphi^* \partial^\alpha \varphi \right) \right]$$

$$= \partial_\mu \left[\frac{\partial}{\partial \partial_\mu \varphi^*} \left(g^{\beta\alpha} \partial_\beta \varphi^* \partial_\alpha \varphi \right) \right]$$

$$= \partial_\mu g^{\alpha\beta} \delta_{\beta\mu} \partial_\alpha \varphi = \partial_\mu g^{\alpha\mu} \partial_\alpha \varphi = \partial^2 \varphi$$

$$\frac{\partial \mathcal{L}}{\partial \varphi^*} = \frac{\partial}{\partial \varphi^*} \left[-m \varphi^* \varphi \right] = -m \varphi$$

$$\Rightarrow \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^*} - \frac{\partial \mathcal{L}}{\partial \varphi^*} = (\partial^2 + m^2) \varphi = 0$$

$\rightarrow$ K.G equation !!

But φ here is a classic field! NOT the probability wave function

nothing to do with quantum at all at the moment!

So $i\partial_t \not\rightarrow \hat{H}$ $\quad -i\partial \not\rightarrow \hat{p}$ No! No! NO!

we have NOT talked about quantization yet.

Let's calculate the Hamiltonian:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = \dot\varphi^* \qquad \mathcal{L} = \dot\varphi^*\dot\varphi - \vec\partial\varphi^*\vec\partial\varphi - m^2\varphi^*\varphi$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot\varphi^*} = \dot\varphi$$

$$\mathcal{H} = \pi\dot\varphi + \pi^*\dot\varphi^* \underbrace{- \dot\varphi^*\dot\varphi + \vec\partial\varphi^*\vec\partial\varphi + m^2\varphi^*\varphi}_{-\mathcal{L}}$$

$$= \dot\varphi^*\dot\varphi + \dot\varphi^*\dot\varphi - \dot\varphi^*\dot\varphi + \vec\partial\varphi^*\vec\partial\varphi + m^2\varphi^*\varphi$$

$$= \pi^*\pi + \vec\partial\varphi^*\vec\partial\varphi + m^2\varphi^*\varphi.$$

Positive definite!!

Consider the transformation

$$\varphi \to \varphi e^{i\alpha}$$
$$\varphi^* \to \varphi^* e^{-i\alpha} \qquad \text{with } \alpha = \text{const.}$$

$$\mathcal{L}' = \partial_\mu\varphi^* \cancel{e^{-i\alpha}}\, \partial_\mu\varphi\, \cancel{e^{i\alpha}} - m^2\varphi^*\cancel{e^{-i\alpha}}\,\varphi\,\cancel{e^{i\alpha}} = \mathcal{L} \quad \text{unchanged}$$

take $\alpha \to 0$, $\quad \varphi \to \varphi(1+i\alpha) \Rightarrow \delta\varphi = i\alpha\varphi.$

$\qquad\qquad\quad \varphi^* \to \varphi^*(1-i\alpha) \Rightarrow \delta\varphi^* = -i\alpha\varphi^*$

$$\alpha\, j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\,\delta\varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^*}\,\delta\varphi^*$$

$$= \partial^\mu \varphi^*\, i\alpha\varphi + \partial_\mu \varphi\,(-i\alpha\varphi^*)$$

$$= \alpha\left[i\varphi\,\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi\right]$$

$$\Rightarrow\ j^\mu \propto i\varphi\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi$$

NOT the probability current! But a charge current!

Why charge? we will see this in the future when we couple φ & φ^* to the E&M vector field A_μ.

example 5. Classic E & M field

$$\mathcal{L}[A_\mu, \partial_\nu A_\mu] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. $\qquad A_\mu \equiv (\Phi, \vec{A})$

- note that we prefer A_μ than $\vec{B}$ and $\vec{E}$ as the fundamental fields in the Lagrangian. Since $\vec{E}$ and $\vec{B}$ can not describe things like A-B effects, but A^μ can. Although A^μ are NOT observables, while $\vec{E}$ and $\vec{B}$ are.

Let's get familiar with $F_{\mu\nu}$

relates to the $\vec{B}$ and $\vec{E}$ fields:

$$\vec{B} \equiv \vec{\nabla}\times\vec{A}\ , \qquad \vec{E} \equiv -\vec{\nabla}\Phi - \dot{\vec{A}}$$

$$F_{0i} = \partial_0 A_i + \partial_i A_0 = -E_i$$

$$F_{ij} = -\partial_i A_j + \partial_j A_i = -\epsilon_{ijk} B_k$$

or $\quad E_i = -F_{0i}\ , \quad \underline{B_i = -\frac{1}{2}\epsilon_{ijk} F_{jk}}$

$$\Rightarrow \quad F_{\mu\nu} = \begin{array}{cc} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

check.

$$\vec{B} = \begin{vmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$B_1 = \partial_2 A_3 - \partial_3 A_2 = -F_{23}$

$F_{23} = -B_1 = -\epsilon_{231} B_1$ ✓

$B_2 = -\partial_1 A_3 + \partial_3 A_1 = F_{13}$

$F_{13} = B_2 = -\epsilon_{132} B_2$ ✓

$B_3 = \partial_1 A_2 - \partial_2 A_1 = -F_{12}$

$F_{12} = -B_3 = -\epsilon_{123} B_3$ ✓

$B_i = \epsilon_{ijk}\partial_j A_k$

$= -\epsilon_{ijk}(-\partial_j A_k)$

$= -\frac{1}{2}\epsilon_{ijk}(-\partial_j A_k + \partial_k A_j)$

$= -\frac{1}{2}\epsilon_{ijk} F_{jk}$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -\frac{1}{4} g^{00} g^{ij} (\partial_0 A_i + \partial_i A_0)(\partial_0 A_j + \partial_j A_0) \times 2$$

$$-\frac{1}{4} g^{il} g^{jm} (-\partial_i A_j + \partial_j A_i)(-\partial_l A_m + \partial_m A_l)$$

$$= -\frac{1}{2}(-\delta^{ij}) E_i E_j - \frac{1}{4} \underbrace{\delta^{il}\delta^{jm}\epsilon_{ijk}\epsilon_{lmn}}_{=2\delta_{kn}} B_k B_n$$

$$= \frac{1}{2}\vec{E}^2 - \frac{1}{2}\vec{B}^2$$

equation of motion (Maxwell eqn.)

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \partial_\mu \frac{\partial}{\partial \partial_\mu A_\nu} \left[g^{\alpha\rho} g^{\beta\sigma} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \right] \left(-\tfrac{1}{4}\right)$$

$$= \partial_\mu \Big(g^{\alpha\rho} g^{\beta\sigma} (\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$+ g^{\alpha\rho} g^{\beta\sigma} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) \Big) \left(-\tfrac{1}{4}\right)$$

$$= \partial_\mu \Big((g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma})(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$+ (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu})(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \Big) \left(-\tfrac{1}{4}\right)$$

$$= \partial_\mu \big(\partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$+ \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu A^\nu - \partial^\nu A^\mu \big) \left(-\tfrac{1}{4}\right)$$

$$= -\partial_\mu F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \qquad \Rightarrow \quad \partial_\mu F^{\mu\nu} = 0$$

plug in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

$$\partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = 0$$

$$\Rightarrow \quad \begin{aligned} &\partial^2 A_0 - \partial_0 (\partial_0 A_0 + \vec{\partial} \cdot \vec{A}) = 0 \\ &\partial^2 A_i + \partial_i (\partial_0 A_0 + \vec{\partial} \cdot \vec{A}) = 0 \end{aligned}$$

$\Rightarrow \quad \vec{\partial}^2 A_0 = -\vec{\partial} \cdot \vec{A}$

A_0 is not dynamic

no $\dot{A}_0$ in the Lagrangian.

$$\Rightarrow \quad -\vec{\partial}^2 A_0 - \partial_0 \vec{\partial}\cdot\vec{A} = -\vec{\partial}\cdot(\vec{\partial} A_0 + \partial_0 \vec{A}) \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$$

$$\partial_0^2 A_i + \partial_i \partial_0 A_0 - \vec{\partial}^2 A_i + \partial_i \vec{\partial}\cdot\vec{A}$$

$$= \partial_0(\partial_0 A_i + \partial_i A_0) + (\vec{\nabla}\times\vec{\nabla}\times\vec{A})_i$$

$$= -\partial_0 E_i + (\vec{\nabla}\times\vec{B})_i \Longrightarrow \quad \vec{\nabla}\times\vec{B} = \partial_0 \vec{E}$$

Only 2 eqns. from the euler-Lagrangian eqn. The form will be modified, If we have charge / current

The other 2 Maxwell eqns will be given by the Bianchi Identity:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

proof

Even with charge, the form will not be changed

$$\partial_\lambda \cancel{\partial_\mu A_\nu} - \partial_\lambda \cancel{\partial_\nu A_\mu} + \partial_\mu \cancel{\partial_\nu A_\lambda} - \partial_\mu \cancel{\partial_\lambda} A_\nu + \partial_\nu \cancel{\partial_\lambda} A_\mu - \partial_\nu \cancel{\partial_\mu} A_\lambda = 0$$

Let $\lambda = 1, \mu = 2, \nu = 3$.

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\Rightarrow \quad \partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12} = 0$$

$$\Rightarrow \quad \frac{1}{2}\epsilon_{ijk}\partial_i F_{jk} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$