– Notations & Units

Metric

$$\begin{array}{c} \text{time} \leftarrow \text{space} \rightarrow \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \end{array}$$

$$g^{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

e.g.

vectors

$$p_1 \cdot p_2 \equiv g^{\mu\nu} p_{1\mu} p_{2\nu} = p_1^0 p_2^0 - p_1^1 p_2^1 - p_1^2 p_2^2 - p_1^3 p_2^3$$

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu \equiv (\partial_0, \vec{\partial}) \quad , \quad p_\mu \equiv (p^0, \vec{p})$$

$$\partial_\mu \equiv (\partial_0, -\vec{\partial}) \leftarrow$$

different sign.

$$\Rightarrow \quad \partial^\mu j_\mu = g^{\mu\nu}\partial_\mu j_\nu = \partial_0 j_0 - (-\vec{\partial}) \cdot \vec{j} = \partial_0 j_0 + \vec{\partial} \cdot \vec{j}$$

$$\partial_\mu \partial^\mu \phi = \partial_0^2 \phi - \vec{\partial}^2 \phi$$

tensors

$$F^{\mu\nu} F_{\mu\nu} = g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}$$

* Units : $c = 3\times 10^8$ m/s

$\hbar = 1.055 \times 10^{-34}$ Joule-s

let's set $c = \hbar = 1$

e.g. $E^2 = p^2c^2 + m^2c^4 = p^2 + m^2$

$[\hat{x}, \hat{p}] = i\hbar = i$

From that, we can see that

[Energy] = [momentum] = [mass]

= [length]$^{-1}$ = [time]$^{-1}$

We use [Energy] = GeV $= 10^9$ eV $= 1.602 \times 10^{-10}$ Joule

some examples :

[Area] = [length]$^{-2}$ = GeV^{-2}

[velocity] = $\frac{\text{[length]}}{\text{[time]}}$ = [1] = [c] = GeV^0

$\int d^3x$ = [length]^3 = GeV^{-3}

Some examples cont.

- $[\psi] = GeV^{3/2}$ since $\int d^3x \, |\psi|^2 = 1$

non-relativistic wave function

d^3x: GeV^{-3} ; 1: GeV^0

- $[S] = GeV^0$ since $\exp[i \frac{S}{\hbar}]$

action

dimensionless

or $S \sim \int dt \, \frac{1}{2} m v^2$

dt: GeV^{-1} ; m: GeV ; v: GeV^0

- then $[L] = GeV$, since $S = \int dt \, L$

Lagrangian

and $[\mathcal{L}] = GeV^4$

(1+3)-D lagrangian density $L = \int d^3x \, \mathcal{L}(x)$

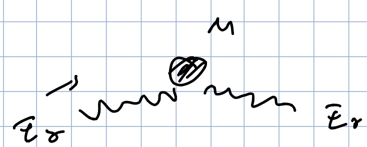
- $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, if $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

$\Rightarrow [F^{\mu\nu}] = GeV^2$

- Dimensional Analysis

- why is sky blue?

we study the elastic scattering of a photon with energy E_γ off an atom with mass M



we consider the case in which

$$E_\gamma \ll \Delta E \ll r_0^{-1} \ll M$$

excitation Energy   Bohr radius   mass

so it is elastic, in the sense that the electron can not be excited by the photon, since $E_\gamma \ll \Delta E$

the photon sees the atom as a whole, it can NOT resolve the atom.

To study the process, we want to write down the Lagrangian following some fundamental rules, out of several building blocks. The kinematic part of the Lagrangian should contain all physics of order E_γ

- building blocks:

  $F_{\mu\nu}$ $(\vec{E}/\vec{B})$ $\rightsquigarrow$ create, annihilate a photon

  ϕ_v $\sim a + a^\dagger$ $\rightsquigarrow$ create, annihilate a atom

  ∂_μ $\sim E_\gamma, q_\gamma$ $\rightsquigarrow$ Energy or momentum of photon

  v_μ $\sim$ 4-velocity of the atom $\rightsquigarrow$ almost at rest
  $v^\mu \sim (1, \vec{0})$

- Fundamental rules:

  * Atom number is conserved.
    — we do not create or destroy an atom in elastic scattering.

    $\Rightarrow$ $\phi_v^\dagger$ & ϕ_v appear in pair!

    No $\phi_v^\dagger \phi_v \phi_v$ term in the Lagrangian is allowed!

  * Charge & Parity conservation

    $\Rightarrow$ No $\varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu}$

* Gauge Invariance. You will learn this in the future

but here All you need to know is that.

- a charge neutral object is Gauge invariant.

so ϕ_v, ϕ_v^+ is. (Atom is charge neutral)

- $F_{\mu\nu}$ is gauge invariant. (also anti-symmetric)

* We to construct out of the building blocks a scalar. since $\mathcal{L}$ is a scalar

$\Rightarrow$ all Lorentz Indices have to be contracted.

So the Lagrangian we can write down is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$ ~ kinematics of photon, not interesting

$$+ C_1 \underbrace{\phi_v^+ \phi_v}_{\text{physics} \sim E_r} F_{\mu\nu} F^{\mu\nu} + C_2 \underbrace{\phi_v^+ \phi_v}_{\text{physics} \sim E_r} V_\mu F^{\mu\nu} V^\alpha F_{\alpha\nu}$$

$$+ C_3 \underbrace{\phi_v^+ \phi_v}_{\text{physics} \sim E_r} \partial_\lambda F^{\mu\nu} F_{\alpha\nu} V^\alpha + \dots$$

interactions between photon and atom

$C_i = C_i (\delta E, . M, r_0^{-1})$ → called wilson coefficients.

Since

$$[\phi_v] = \text{GeV}^{3/2}, \; [F^{\mu\nu}] = \text{GeV}^2 . \; [\mathcal{L}] = \text{GeV}^4$$

$\Rightarrow$ $[C_1] = [C_2] = \text{GeV}^{-3}$. $[C_3] = \text{GeV}^{-5}$

$\Rightarrow \quad c_1 \sim \left(\frac{1}{\Delta E}\right)^3, \quad c_2 \sim \left(\frac{1}{\Delta E}\right)^3, \quad [c_3] \sim \left(\frac{1}{\Delta E}\right)^5$

$$\Rightarrow \quad c_1 \phi_v^+ \phi_v F_{\mu\nu} F^{\mu\nu} \sim \frac{E_\gamma^7}{\Delta E^3} \sim \left(\frac{E_\gamma}{\Delta E}\right)^3 E_\gamma^4$$

$$c_2 \phi_v^+ \phi_v v_\mu F^{\mu\nu} F_{\alpha\nu} v^\alpha \sim \frac{E_\gamma^7}{\Delta E^3} \sim \left(\frac{E_\gamma}{\Delta E}\right)^3 E_\gamma^4$$

$$c_3 \phi_v^+ \phi_v \partial_\mu F^{\mu\nu} \partial^\alpha F_{\alpha\nu} \sim \frac{E_\gamma^9}{\Delta E^5} \sim \left(\frac{E_\gamma}{\Delta E}\right)^5 E_\gamma^4$$

since $E_\gamma \ll \Delta E$. $\Rightarrow$ c_3 term can be negligible when compared with c_1 & c_2 term.

the cross section is proportional to the transition amplitude^2

$$\underbrace{\sigma}_{\text{cross-section}} = |\langle \underbrace{A}_{\text{atom}} \underbrace{\gamma}_{\text{photon}} | \mathcal{L}_{int} | A\gamma \rangle|^2 \cdot \text{phase space factor of photon}$$

$$= |\langle A\gamma | c_1 \cdots + c_2 \cdots | A\gamma \rangle|^2 \cdot \text{phase space factor of photon}$$

$$= \Big[\underbrace{|c_1|^2}_{\left(\frac{1}{\Delta E}\right)^6} + \underbrace{|c_2|^2}_{\left(\frac{1}{\Delta E}\right)^6} + \underbrace{(c_1^* c_2 + c.c.)}_{\left(\frac{1}{\Delta E}\right)^6}\Big] \cdot \text{phase space factor of photon}$$

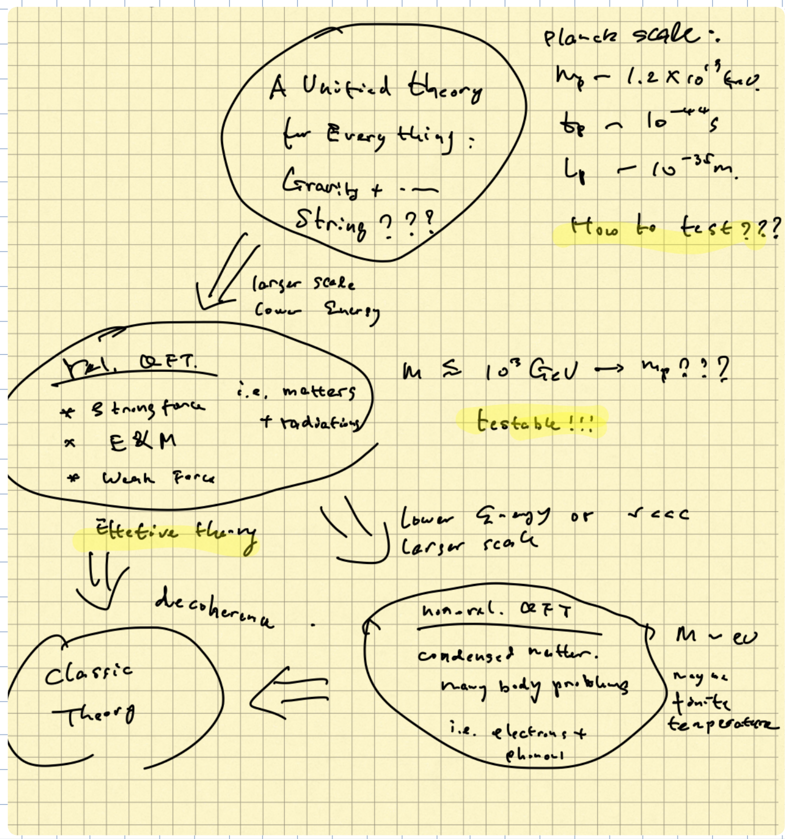
since $[\sigma] = [\text{length}]^{-2} = \text{GeV}^{-2}$

$\Rightarrow \quad \sigma = \frac{E_\gamma^4}{\Delta E^6}$ to make the dimension correct!

$\rightarrow \quad \sigma \propto \frac{1}{\lambda^4}$ <— releigh law w/o doing any calculation!!!

- why we study QFT ?

We are here and below



- QFT can NOT be valid to arbitrarily high Energy Scale.
- It is an Effective theory for $E \lesssim 10^3$ TeV or even higher.
- QFT is NOT necessarily relativistic
- relativity + QM $\Rightarrow$ Many body $\Rightarrow$ Field Theory.

- Early Attempts: relativity + QM.

  * Klein-Gordon Equation

    first attempt to combine QM with special Rel.

    Discovered first by Shrödinger. before Shrödinger Eqn.

    but abandoned by him for can not describe the hydrogen fine structure.

Recall QM: Shrödinger Eqn.

$$E\psi = \frac{p^2}{2m}\psi$$

$$E \to i\partial_t , \quad p \to -i\vec{\partial}$$

$$\Rightarrow \quad i\partial_t \psi = -\frac{\vec{\partial}^2}{2m}\psi$$

ψ: wave function for a single particle!

$$\Rightarrow \quad i\psi^*\partial_t\psi = -\psi^*\frac{\vec{\partial}^2}{2m}\psi \qquad ①$$

$$-i\psi\,\partial_t\psi^* = -\psi\frac{\vec{\partial}^2}{2m}\psi^* \qquad ②$$

$$① - ② \quad i\partial_t|\psi|^2 = \psi\frac{\vec{\partial}^2}{2m}\psi^* - \psi^*\frac{\vec{\partial}^2}{2m}\psi$$

$$\Rightarrow \quad i\partial_t|\psi|^2 = \vec{\partial}\cdot\left(\psi\frac{\vec{\partial}}{2m}\psi^* - \psi^*\frac{\vec{\partial}}{2m}\psi\right)$$

$$\Rightarrow \quad \partial_t|\psi|^2 = \vec{\nabla}\cdot\left(i\psi^*\frac{\vec{\partial}}{2m}\psi + c.c\right)$$

$$\Rightarrow \quad \partial_t|\psi|^2 = -\vec{\nabla}\cdot\left(\psi^*\frac{\vec{\partial}}{2mi}\psi + c.c.\right)$$

$$\Rightarrow \quad \partial_t\rho = -\vec{\nabla}\cdot\vec{j} \quad \text{or} \quad \partial_\mu j^\mu = 0$$

$$j_\mu \equiv (\rho, \vec{j}) , \quad \rho = |\psi|^2 > 0$$

Now to construct a relativistic QM

natural attempts to use

$$E^2 = p^2 + m^2$$

$$\Rightarrow \quad E^2 \phi = (\vec{p}^2 + m^2)\phi$$

$$\Rightarrow \quad (i\partial_t)^2 \phi = \left[(-i\vec{\partial})^2 + m^2\right]\phi$$

$$\Rightarrow \quad -\partial_t^2 \phi = (-\vec{\partial}^2 + m^2)\phi$$

$$\Rightarrow \quad \partial_t^2 \phi = \vec{\partial}^2 - m^2 \phi$$

or

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

Gives the Klein-Gordon Eqn.

Again, ϕ is the wave function for a single particle!

Now we study the probability current.

$$\phi^* \partial_t^2 \phi = \phi^* (\vec{\partial}^2 - m^2)\phi \qquad (1)$$

$$\phi \, \partial_t^2 \phi^* = \phi \, (\vec{\partial}^2 - m^2)\phi^* \qquad (2)$$

$$(1)-(2) \quad \phi^* \partial_t^2 \phi - \phi \partial_t^2 \phi^* = \phi^* \vec{\partial}^2 \phi - \phi \vec{\partial}^2 \phi^*$$

$$\Rightarrow \quad \partial_t (\phi^* \partial_t \phi - \phi \partial_t \phi^*) = \vec{\partial} \cdot (\phi^* \vec{\partial} \phi - \phi \vec{\partial} \phi^*)$$

$$\Rightarrow \quad \partial_t \left(\phi^* \frac{\partial_t \phi}{-2mi} + c.c.\right) = -\vec{\partial} \cdot \left(\phi^* \frac{\vec{\partial}}{2mi} \phi + c.c.\right)$$

Now if $\rho = i\phi^* \frac{\partial_t}{2m} \phi + c.c.$

$\vec{j} = \phi^* \frac{\vec{\partial}}{2mi} \phi + c.c.$ or $j_\mu = (\rho, \vec{j})$

Again we have the conservation of the prob. current

$$\partial_t \rho = - \vec{\partial} \cdot \vec{j} \quad \text{or} \quad \partial_\mu j^\mu = 0$$

But

Issue 1. ρ can be negative !?

since $\phi \propto e^{\pm i E t}$, $E = \sqrt{p^2 + m^2} > 0$

$$\rho \propto \pm E |\phi|^2$$

negative probability ???

Can be overcome by interpreting ρ as a charge density

though not good, but acceptable

so negative prob. problem solved

Issue 2. Fail to explain the fine structure of the hydrogen atom

Solving KG equation for H atom. (first done by Shrödinger)

canonical momentum $i\partial_t \to i\partial_t - e\phi$ required by Gauge Sym,

$-i\vec{\partial} \to -i\vec{\partial} - e\vec{A}$ formulated by Weyl.

For H atom. $\phi = -\frac{Ze}{4\pi r}$, $\vec{A} = 0$

$$\Rightarrow \quad \left(i\partial_t + \frac{Z\alpha}{r}\right)^2 \phi = -\vec{\nabla}^2\phi + m^2\phi \qquad \alpha \equiv \frac{e^2}{4\pi}$$

fine structure constant

looking for stationary solution

$$\phi = \tilde{\phi}_\nu \, e^{-iE_\nu t}$$

$$\Rightarrow \quad \left(E_\nu + \frac{Z\alpha}{r}\right)^2 \tilde{\phi}_\nu = -\vec{\nabla}^2\tilde{\phi}_\nu + m^2\tilde{\phi}_\nu$$

$$\Rightarrow \quad \left(E_\nu^2 + 2E_\nu\frac{Z\alpha}{r} + \frac{Z^2\alpha^2}{r^2}\right)\tilde{\phi}_\nu = -\vec{\nabla}^2\tilde{\phi}_\nu + m^2\tilde{\phi}_\nu$$

$$\Rightarrow \quad \left\{\frac{1}{2E_\nu}\left(-\vec{\nabla}^2 - \frac{Z^2\alpha^2}{r^2}\right) - \frac{Z\alpha}{r}\right\}\tilde{\phi}_\nu = \frac{E_\nu^2 - m^2}{2E_\nu}\tilde{\phi}_\nu$$

let $\tilde{\phi}_\nu = R_\nu(r)\, Y_{\ell m}(\theta,\varphi)$, $\ell = 0, 1, 2, 3, \ldots$

$$\Rightarrow \quad \left\{\frac{1}{2E_\nu}\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\overset{\rightsquigarrow\, \lambda(\lambda+1)}{\ell(\ell+1) - Z^2\alpha^2}}{r^2}\right) - \frac{Z\alpha}{r}\right\} R_\nu = \overset{\rightsquigarrow\, E_\nu}{\frac{E_\nu^2 - m^2}{2E_\nu}} R_\nu \qquad \textcircled{\small D}$$

Compare with the Shrödinger Eqn. in the Columb potential :

$$\left\{ \frac{1}{2m}\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2}\right) - \frac{Z\alpha}{r} \right\} R_n = E_n R_n \qquad (2)$$

if we make the replacement

$2E_\nu \to 2m, \quad \lambda(\lambda+1) \to l(l+1), \quad \epsilon_\nu \to E_n$

(1) becomes (2)

$\Rightarrow$ from the solution of Eqn. (2)

$$\epsilon_\nu = -\frac{1}{2}m\frac{Z^2\alpha^2}{\nu^2}, \quad \nu = \lambda+1, \lambda+2, \lambda+3, \cdots$$

with $\lambda = \left((l+1/2)^2 - Z^2\alpha^2\right)^{1/2} - 1/2$

$$= l - \frac{Z^2\alpha^2}{2l+1} + \cdots$$

$\leftarrow$ relativistic corrections. $v \sim \alpha$

Therefore

$$\epsilon_\nu = \frac{E_\nu^2 - m^2}{2E_\nu} = -\frac{1}{2}m\frac{Z^2\alpha^2}{\nu^2}$$

$$\Rightarrow \quad E_\nu = \pm m\left(1 + \frac{Z^2\alpha^2}{\nu^2}\right)^{-1/2} \quad \cdots \quad (3)$$

$$\nu = \lambda+1, \lambda+2, \cdots$$

$$\Rightarrow \quad \nu = n - \frac{Z^2\alpha^2}{2l+1} + \cdots$$

$\leftarrow$ relativistic corrections

expand ③ in terms of α gives.

$$E_n = \pm m\left[1 - \frac{1}{2}\frac{Z^2\alpha^2}{n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{l+1/2} - 3/4\right) + \cdots\right]$$

negative energy solution

rest energy

$v \ll 1$ result reproduce Shrödinger Eqn.

relativistic corrections

issue a. negative energy solution !

This is really an issue for relativistic QM.

issue b. experimentally, the hydrogen Energy level is well described by

$$E_n = m\left(1 - \frac{1}{2}\frac{Z^2\alpha^2}{n^2} - \frac{Z^4\alpha^4}{n^4}\left(\frac{n}{l+1} - 3/4\right) + \cdots\right)$$

this tiny difference makes people abandon the K.G. Eqn. for a while

2. Dirac Equation

still for a single free particle

To construct an equation

first order in the time derivative ∂_t (Energy)

Special Relativity $\Longrightarrow$ first order in $\vec{\partial}$. (momentum)

propose:

$$i\partial_t \psi = p_0 \psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi$$

$$p_0^2 \psi = \left[(\alpha_i p_i \alpha_j p_j + (\alpha_i \beta + \beta \alpha_i) p_i m + \beta^2 m^2\right]\psi$$
$$= (p^2 + m^2)\psi$$

Hence

$$\begin{cases} \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \\ \alpha_i \beta + \beta \alpha_i = 0 \\ \beta^2 = 1 \end{cases}$$

$$\alpha_i = \begin{bmatrix} -\sigma_i & 0 \\ 0 & +\sigma_i \end{bmatrix}_{4\times 4} \qquad \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{4\times 4} \quad \cdots \text{①}$$

$\sigma_i \equiv$ pauli matrices (2×2)

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}_{2\times 2} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2}$$

- 2×2 is not possible due to the additional β matrix
- 4×4 is the minimum dimension for Dirac Equation
- other representations of α^i and β are possible and are related to eqn. ① by "rotation".

rewrite $\quad p_0 \beta \psi = (\beta \vec{\alpha} \cdot \vec{p} + \beta^2 m) \psi$

$$\Rightarrow (p_0 \gamma^0 - \gamma_i p_i - m) \psi = 0$$

$$\Rightarrow (\gamma^\mu p_\mu - m) \psi = 0$$

$$\Rightarrow (\not{p} - m) \psi = 0$$

$$\Rightarrow (i\not{\partial} - m) \psi = 0$$

$\gamma_0 = \beta \qquad \gamma_i = \beta \vec{\alpha} \qquad \not{p} = \gamma^\mu p_\mu$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma \\ \bar{\sigma} & 0 \end{pmatrix}_{4\times4} \qquad \sigma_\mu = (1, \vec{\sigma})_{2\times2} \qquad \mu = 0, 1, 2, 3$$

$$\bar{\sigma}_\mu = (1, -\vec{\sigma})_{2\times2}$$

$$\sigma_0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}_{2\times2}$$

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

note that

$$\{\gamma_\mu . \gamma_\nu\} = 2 g_{\mu\nu}$$ easy to see from the definition of $\vec{\alpha}$, β

- Solves the negative probability problem.

$$\rho = |\psi|^2 \qquad \vec{j} = \psi^\dagger \vec{\alpha} \psi.$$

Derivation

$$(i\partial_t \gamma_0 \psi + i\vec{\gamma}\cdot\vec{\partial} - m)\psi = 0$$

$$\Rightarrow \quad (i\,\partial_t \psi + i\gamma_0\vec{\gamma}\cdot\vec{\partial} - m\gamma_0)\psi = 0$$

$$i\,\partial_t \psi_i = (-i\gamma_0\vec{\gamma}\cdot\vec{\partial} + m\gamma_0)_{ij}\psi_j$$

$$i, j = 1, \cdots, 4 \qquad \psi = \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix}$$ 4-solutions !!!

$$i\,\psi_i^* \partial_t \psi_i = \psi_i^* (-i\gamma_0\vec{\gamma}\cdot\vec{\partial} + m\gamma_0)_{ij}\psi_j$$

$$-i\,\psi_i\,\partial_t \psi_i^* = \psi_i (i\gamma_0\vec{\gamma}\cdot\vec{\partial} + m\gamma_0)_{ji}\psi_j^*$$

$$i(\psi_i^* \partial_t \psi_i + \psi_i \partial_t \psi_i^*) = -i\left[\psi_i^* (\gamma_0\vec{\gamma})_{ij}\cdot(\vec{\partial}\psi)_j\right] - i\left[(\vec{\partial}\psi^*)_j (\gamma_0\vec{\gamma})_{ji}\psi_i\right]$$

$$i\,\partial_t(\psi_i^*\psi_i) = -i\,\vec{\partial}\cdot\left[\psi_i^*(\gamma_0\vec{\gamma})_{ij}\psi_j\right]$$

$$\partial_t(\psi^\dagger\psi) = -\vec{\partial}\cdot\left[\psi^\dagger\gamma_0\vec{\gamma}\psi\right]$$

$\rho = \psi^\dagger\psi > 0$, $\quad \vec{j} = \psi^\dagger\gamma_0\vec{\gamma}\psi = \psi^\dagger\vec{\alpha}\psi$

rewrite

$$\rho = \psi^{+} \gamma_0 \gamma_0 \psi \equiv \bar{\psi} \gamma_0 \psi \,, \quad \bar{\psi} \equiv \psi^{+} \gamma^{0}$$

$$\vec{j} = \psi^{+} \gamma_0 \vec{\gamma} \psi = \bar{\psi} \vec{\gamma} \psi .$$

Then

$$j_\mu \equiv \bar{\psi} \gamma_\mu \psi \,, \quad \partial^\mu j_\mu = 0$$

– Solves the Hydrogen fine structure problem

Dirac eqn. predicts

$$E_n = \pm m \left(1 - \frac{1}{2} \frac{Z^2 \alpha^2}{n^2} - \frac{Z^4 \alpha^4}{2 n^4} \left(\frac{n}{j + 1/2} - \frac{4}{3} \right) + \cdots \right)$$

with $\quad j = l + 1/2$

↳ due to spin 1/2

but the negative Energy problem remains !

negative energy prob.

For simplicity, we consider a particle at rest :

$\vec{p}=0 \Rightarrow -i\vec{\partial}\psi = 0$

The Dirac eqn. gives

$$i\partial_t \psi = m\gamma^0\psi \,, \quad \text{note } \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{4\times4}$$

$$i\partial_t \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix}_{4\times4} \cdot \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$$

$$i\dot{\psi}_+ = m\psi_- \,, \quad i\dot{\psi}_- = m\psi_+$$

$$i\partial_t(\psi_+ + \psi_-) = m(\psi_+ + \psi_-) \rightarrow 2 \text{ positive energy}$$

$$i\partial_t(\psi_+ - \psi_-) = -m(\psi_+ + \psi_-) \rightarrow 2 \text{ negative energy}$$

In General, we have 4 solutions for Dirac eqn.

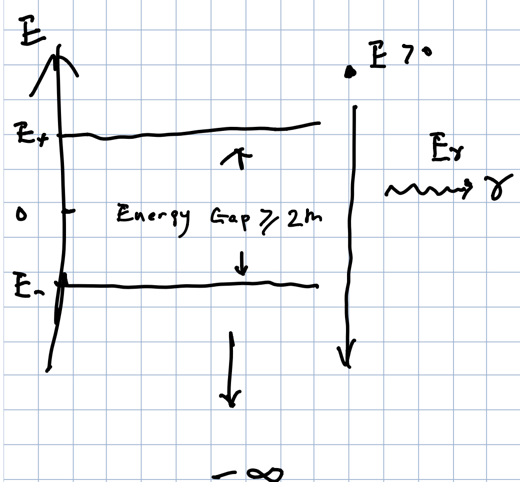
- 2 positive energy solutions with spin $+\frac{1}{2}$ and $-\frac{1}{2}$

$$\psi \propto u^{\uparrow}(p)\, e^{ip\cdot x}, \quad u^{\downarrow}(p)\, e^{ip\cdot x}, \quad E_+ = +\sqrt{p^2+m^2}$$

- 2 negative energy solutions with spin $+\frac{1}{2}$ and $-\frac{1}{2}$

$$\psi \propto \bar{v}^{\uparrow}(p)\, e^{-ip\cdot x}, \quad \bar{v}^{\downarrow}(p)\, e^{-ip\cdot x}, \quad E_- = -\sqrt{p^2+m^2}$$

E_+ E_- 1 to 1 correspondence



- Okay for free particle. Since energy is conserved, the positive energy will stay positive

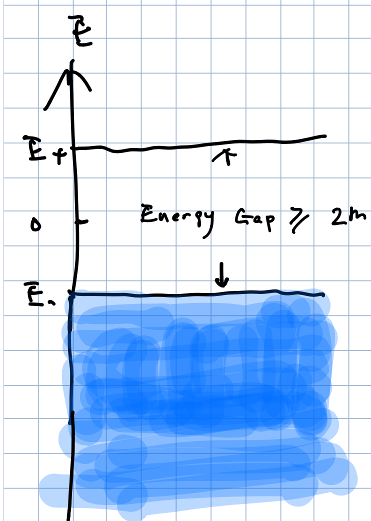
- When interact with E&M radiations $E_+ \xrightarrow{\text{can}} E_-$ by emitting photons !!

- No lower bound. Everything will be UNSTABLE!

A solution proposed by Dirac (1930) :

- abandon the single particle picture

- Dirac Sea (Infinite degrees of freedoms!)



- vaccuum: ($E=0$, $e=0$, for vaccuum)

  all negative energy states are filled with electrons

  all positive energy states are empty

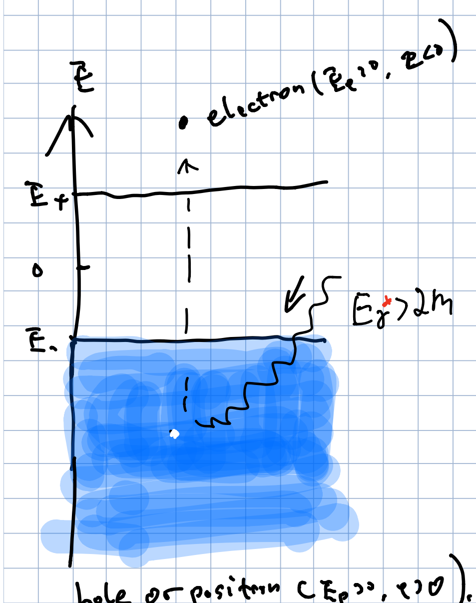
- excitations : ($E>0$, $e<0$)

  put electrons in the positive energy states

- pauli exclusion principle forbids the electrons in the positive energy states fall into the negative energy states.

NOT single particle anymore!!

problem solved!



γ^* excites a negative energy electron ($-E$) into the positive energy state E_e

$$E_e = E_{\gamma^*} - E \ , \quad -e<0$$

also creates a hole E_p

$$E_p = 0-(-E) = E, \quad e$$

positive energy positive charge

predicts the positron!! (1933)

and particles can be created!!!

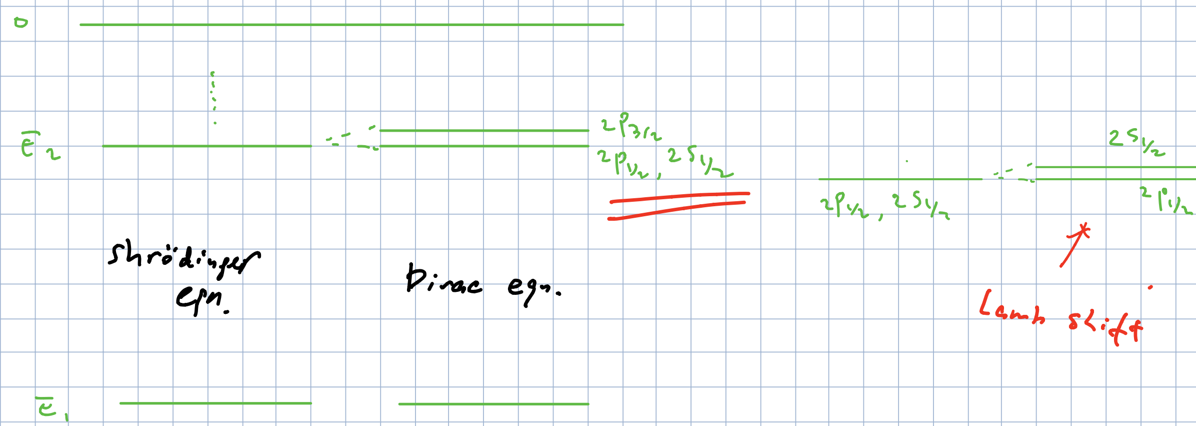
Still problematic!

- Dirac's solution is only valid for spin-½.

Does NOT work for spin-0, spin-1 (photon)

Since NO pauli exclusion principle for spin-0, spin-1

- Hydrogen Spectrum with more accuracy



Dirac: $2P_{1/2}$: $\vec{l} = 1$, $\vec{s} = ½$ $\quad j = ½$ $\qquad nl_j$ notation.

$2S_{1/2}$: $\vec{l} = 0$, $\vec{s} = ½$ $\quad j = ½$

$\rightarrow \Delta E = 0$, degenerate

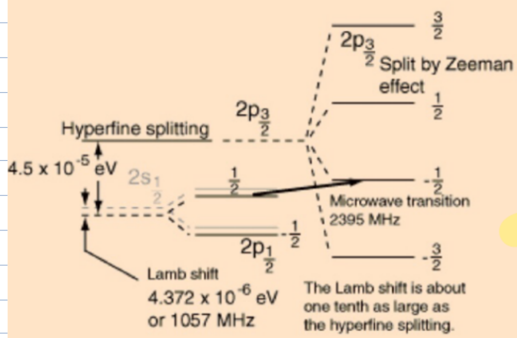
disagree!!

exp: lamb shift (1947) $\rightarrow \Delta E \neq 0$

$\Delta E \simeq 1057$ MHz $\sim 4.4 \times 10^{-6}$ eV

Measurement of the Lamb Shift

Vhile the Lamb shift is extremely small and difficult to measure as a splitting in the ptical or UV spectral lines, it is possible to make use of transitions directly between the ublevels by going to other regions of the electromagnetic spectrum. Willis Lamb made is measurements of the shift in the microwave region. He formed a beam of hydrogen toms in the 2s(1/2) state. These atoms could not directly take the transition to the 1s(1/2) tate because of the selection rule which requires the orbital angular momentum to change y 1 unit in a transition. Putting the atoms in a magnetic field to split the levels by the Zeeman effect, he exposed the atoms to microwave radiation at 2395 MHz (not too far rom the ordinary microwave oven frequency of 2560 MHz).



Then he varied the magnetic field until that frequency produced transitions from the 2p(1/2) to 2p(3/2) levels. He could then measure the allowed transition from the 2p(3/2) to the 1s(1/2) state. He used the results to determine that the zero-magnetic field splitting of these levels correspond to 1057 MHz. By the Planck relationship, this told him that the energy separation was 4.372 E-6 eV.