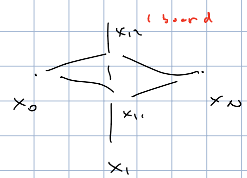
Some comments on the previous lectures:

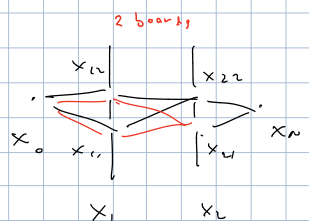
1 $\quad N\int \mathcal{D}[x] \equiv \lim_{N\to\infty} \left(\frac{1}{2\pi i\hbar\epsilon}\right)^{N/2} \int \prod_{i=1}^{N-1} dx_i$

product

2. why product?



$\langle x_N | x_{11}\rangle \langle x_{11} | x_0\rangle + \langle x_N | x_{12}\rangle \langle x_{12} | x_0\rangle$



$\langle x_N | x_{21}\rangle$

$\left(\langle x_{21} | x_{11}\rangle \langle x_{11} | x_1\rangle + \langle x_{21} | x_{12}\rangle \langle x_{12} | x_0\rangle \right)$

$+ \langle x_N | x_{22}\rangle$

$\left(\langle x_{22} | x_{11}\rangle \langle x_{11} | x_1\rangle + \langle x_{22} | x_{12}\rangle \langle x_{12} | x_0\rangle \right)$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} \langle x_N | x_{2i}\rangle \langle x_{2i} | x_{1j}\rangle \langle x_{1j} | x_0\rangle$$

$\xrightarrow{N-1 \text{ boards}}$ $\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_{N-1}=1}^{2} \langle x_N | x_{i_{N-1}}\rangle \langle x_{i_{N-1}} | x_{i_{N-2}}\rangle \cdots \langle x_{i_1} | x_0\rangle$

$\xrightarrow{\text{infinit holes}}$ $\int dx_1\, dx_2 \cdots dx_{N-1}\, \langle x_N | x_{N-1}\rangle \langle x_{N-1} | x_{N-2}\rangle \cdots \langle x_1 | x_0\rangle$

2. analytic continuation

$$F[t],\ t\in\mathbb{R} \xrightarrow{\text{analytic cont.}} F[t],\ t\in\mathbb{C}^*$$

only allowed when $F[t]$ is well-defined on $t\in\mathbb{C}^*$

Note $F[t],\ t\in\mathbb{R} \neq F[t],\ t\in\mathbb{C}$

since they are defined on different domain

though formally they are the same.

Then why we want to do analytic continuation?

1. make $F[t]$ well-defined

e.g. $\int_0^\infty dt\, e^{it\alpha}$ is not well-defined at all

but $\int_0^{+\infty} dt\, e^{-t\alpha}$ is well-defined

Crucial for numerical evaluation!!! Do calculation after analytic continuation then turn back!

2. use to extract parameters.

e.g. $e^{iE_0 t} = g(E_0, t)$,

can be used to solve for E_0.

Same E_0!! E_0 is not changed

Then $E^{-E_0 \tau} = g(E_0, \tau)$

So can also be used to solve for E_0

3. better features

e.g.

highly oscillating

$$e^{iE_0 t} + e^{iE_1 t} + e^{iE_2 t} = g(E_0\ t) \qquad E_0 < E_1 < E_2 \cdots$$

Then after analytic continuation.

$$e^{-E_0\tau} + e^{-E_1\tau} + \cdots = g(E_1 \cdots E_0\ \tau)$$

$$\xrightarrow{\tau\to\infty} e^{-E_0\tau} = g(E_0\ \tau)$$

Last class:

$$\mathcal{N}\int \mathcal{D}[q(\tau)]\, e^{-S_E} \simeq \mathcal{N}\, e^{-S_E(\bar{q})}\, \mathrm{Det}\left[-\partial_\tau^2 + V''(\bar{q})\right]^{-1/2} + \mathcal{O}(\hbar^3)$$

Semi-classic Approximation

To obtain this, we expand S_E around $\bar{q}$

$$S_E(q) = S_E(\bar{q}) + \delta S$$

$$= \underbrace{S_E(\bar{q})}_{\text{semi-classical minimum of } S_E(q)\,!!} + \cancel{\left.\frac{\delta S_E}{\delta q}\right|_{\bar{q}} \delta q}^{\,0} + \underbrace{\left.\frac{\delta^2 S_E}{\delta^2 q}\right|_{\bar{q}} (\delta q)^2}_{\text{Quantum corrections to } S_E(\bar{q})} + \ldots$$

and $\delta S = \dfrac{\delta^2 S_E}{\delta^2 q}(\delta q)^2 = \displaystyle\int d\tau\, \tfrac{1}{2}\delta q \left(-\partial_\tau^2 + V''(\bar{q})\right)\delta q$

$$\delta q = \sum_n C_n \psi_n(\tau)$$

where

$$\left(-\partial_\tau^2 + V''(\bar{q})\right)\psi_n(\tau) = \epsilon_n \psi_n(\tau)$$

← compare with Shrödinger equation.

$$\left(-\frac{\nabla^2}{2m} + U(q)\right)\psi_n(q) = \epsilon_n \psi_n(q)$$

So

$$\underline{-\frac{\nabla^2}{2m} \to \partial_\tau^2, \quad U(q) \to V''(\bar{q})}$$

$$\Rightarrow \quad \delta S = \sum_n \frac{1}{2}\epsilon_n \underbrace{C_n^2}_{\text{pretty much the corrections to the action !!}}$$

Harmonic oscillator

if $x_i = x_f = 0 \Rightarrow \bar{x} = 0, \quad S[\bar{x}] = 0$

$$\left(-\frac{\partial^2}{\partial \tau^2} + \omega^2\right)\delta x_i = \epsilon_i \delta x_i$$

*compare with $-\frac{\partial^2}{2m}\psi + V(x)\psi = E\psi$

$$\delta x_n \propto \sin \frac{n\pi}{T}\tau \quad \text{or} \quad \cos \frac{n\pi}{T}\tau$$

$$\Rightarrow \quad \epsilon_n = \frac{n^2\pi^2}{T^2} + \omega^2 \qquad n = 0, 1, 2 \ldots$$

since $\tau = 0$ $\delta x = 0$
$\tau = T$ $\delta x = 0$

$$\Rightarrow \quad e^{-S[\bar{x}]} N \cdot \prod_n \left(\frac{n^2\pi^2}{T^2} + \omega^2\right)^{-1/2}$$

$$= e^{-S[\bar{x}]} \underbrace{N \prod_n \left(\frac{n^2\pi^2}{T^2}\right)^{-1/2}}_{\omega = 0, \text{ free particle}} \cdot \prod_n \left(1 + \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-1/2}$$

$$N \prod_n \left(\frac{n^2\pi^2}{T^2}\right)^{-1/2} = \langle x_f{=}x | e^{-\frac{p^2}{2}T} | x_i{=}x \rangle = \sum_n \int \frac{dp_n}{2\pi} \langle x | p_n \rangle \langle p_n | x \rangle e^{-\frac{p_n^2}{2}T}$$

$$= \sum_n \int \frac{dp_n}{2\pi} e^{-\frac{p_n^2}{2}T} = \frac{1}{\sqrt{2\pi T}}$$

Reproduce our text book results for free particle!!

use

$$\prod_n \left(1 + \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-1/2} = \left(\frac{1}{\omega T}\sinh \omega T\right)^{-1/2}$$

$$\Rightarrow \quad N \prod_n \left(\frac{n^2\pi^2}{T^2}\right)^{-1/2} \cdot \prod_n \left(1 + \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-1/2}$$

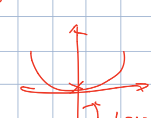
$$= \frac{1}{\sqrt{2\pi T}} \left[\frac{1}{\omega T}\sinh \omega T\right]^{-1/2}$$

nothing but $x_i = x_f = 0$ results
since $S_E[0] = 0$

$$= \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \left[1 + \frac{1}{2} e^{-2\omega T} + \cdots\right]$$

Good enough for ground states since

$|\psi_0(0)|^2$ Ground Energy $n = 2$



lowest Energy when $x = 0$

For General x

$$-\frac{d^2\bar{x}}{d\tau^2} + U'(\bar{x}) = 0 \qquad -\frac{d^2\bar{x}}{d\tau^2} + \omega^2\bar{x} = 0$$

$$\bar{x} = A\exp\{-\omega\tau\} + B\exp[\omega\tau]$$

$$\bar{x}_i = A\exp[0] + B[0] = x$$

$$\bar{x}_f = A\exp[-\omega T] + B\exp[\omega T] = x$$

$$A + B = x$$

$$A\exp[-\omega T] + B\exp[\omega T] = x$$

$$A\left[\exp[\omega T] - \exp[-\omega T]\right) = x\left(\exp[\omega T] - 1\right) \;]$$

$$A = \frac{x\left[\exp[\omega T] - 1\right.}{\exp[\omega T] - \exp[-\omega T]} \xrightarrow{T\to\infty} x$$

$$B = -\frac{x\left[\exp[-\omega T] - 1\right]}{\exp[+\omega T] - \exp[\omega T]} \xrightarrow{T\to\infty} 0$$

$\Rightarrow$

$$\bar{x} = A\exp[-\omega\tau] + B\exp[\omega\tau]$$

$$\frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2 = \left(-A\omega\exp[-\omega\tau] + B\omega\exp[\omega\tau]\right)^2$$

$$= A^2\omega^2\exp[-2\omega\tau] + B^2\omega^2\exp[2\omega\tau] - 2AB\omega^2$$

$$V = \frac{1}{2}\omega^2\left[A^2\exp[-2\omega\tau] + B^2\exp[2\omega\tau] + 2AB\omega^2\right.$$

$$\Rightarrow \quad L = \omega^2 A^2\exp[-2\omega\tau] + \omega^2 B^2\exp[2\omega\tau]$$

$$S[\bar{x}] = \int_0^T d\tau\, L = \omega^2 A^2\left[\frac{-1}{2\omega}\right]\left[\exp[-2\omega T] - 1\right]$$

$$+ \omega^2 B^2\left[\frac{1}{2\omega}\right]\left[\exp[2\omega T] - 1\right]$$

$$= \omega x^2\,\mathrm{Tanh}\left[\frac{\omega T}{2}\right]$$

$\Rightarrow$

$$\int \rho[x]\, e^{-S_E} = \exp[-\omega x^2 \tanh \tfrac{\omega T}{2}]\ \frac{1}{\sqrt{2\pi T}} \left[\frac{1}{\omega T} \sinh \omega T\right]^{-1/2}$$

$$= \sqrt{\frac{\omega}{\pi}}\, \exp[-\omega x^2 \cdot \tanh \tfrac{\omega T}{2}]\ \exp[-\tfrac{\omega}{2} T]\ (1-\exp[-2\omega T])^{1/2}$$

Remind you of the equivalence between these two !!

$$= \sqrt{\frac{\omega}{\pi}}\, \exp(-z]\quad \exp[-\tfrac{\omega}{2} T]$$

$$\times \left(1 + 2z \exp[-\omega T] + \tfrac{1}{2}(-1-4z+4z^2)\exp[-2\omega T] + \cdots\right)$$

$z = \omega x^2$

$$= \sum_n |\varphi_n(x)|^2 \exp[-E_n T]$$

CHECK

$$\varphi_n = \frac{1}{\sqrt{2^n n!}} \left(\frac{\omega}{\pi}\right)^{1/4} \exp[-\tfrac{z}{2})\ H_n(\sqrt{z})$$

$$|\varphi_n|^2 = \frac{1}{2^n n!} \left(\frac{\omega}{\pi}\right)^{1/2} \exp[-z)\ H_n(\sqrt{z})\, H_n^*(\sqrt{z})$$

$n=0$ $\quad \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-z)$ AGREE !

$n=1$ $\quad \frac{1}{2}\left(\frac{\omega}{\pi}\right)^{1/2} \exp(-z)\ 2^2 \sqrt{z}^2 = \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-z)\, 2z$ AGREE !

$n=2$ $\quad \frac{1}{4}\,\frac{1}{2}\left(\frac{\omega}{\pi}\right)^{1/2} \exp(-z)\ (4z-2)^2$

$$= \frac{1}{2}\left(\frac{\omega}{\pi}\right)^{1/2} \exp(-z)\ (-1 + 4z^2 - 4z)$$

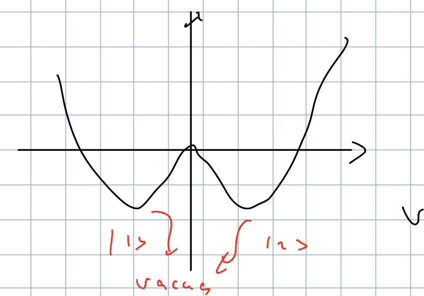
AGREE !

Now double well.

$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!}q^4$

$\to L_E = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \frac{\lambda}{4!}q^4$

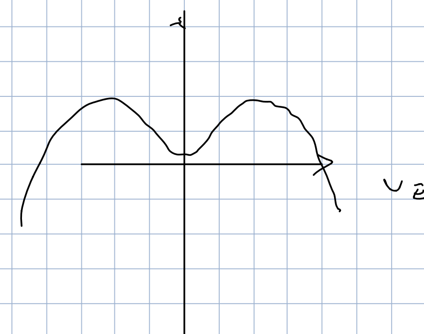
* perturbation in λ does NOT work. perturbation in one of the valley gives degenerate ground states while true ground state is non degenerate. easy to see with a simple model:



$H|1\rangle = E|1\rangle + \Delta|2\rangle$

$H|2\rangle = E|2\rangle + \Delta|1\rangle$

$\Rightarrow H = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$



$\Rightarrow E_+ = E + \Delta, \quad E_- = E - \Delta$

non-degenerate ground state

What perturbation missed is the barrier penetration

best guess for E is $\frac{1}{2}\omega$

solve Δ using Feynman Path Integral

$$\langle x_f | e^{-ET} | x_i \rangle = \int D[x]\, e^{-S_E} = N e^{-\bar{S}_E} \det[-\partial_\tau^2 + V(\bar{x})]^{-1/2}$$

$$V(q) = \frac{\lambda}{4!}(q^2 - \eta^2)^2 \quad \text{or} \quad V_E(q^2) = -\frac{\lambda}{4!}(q^2-\eta^2)^2, \quad \eta^2\lambda = 6\omega^2$$

$$L_E = \frac{1}{2}\dot{q}^2 - V_E(q) = \frac{1}{2}\dot{q}^2 + \frac{\lambda}{4!}(q^2-\eta^2)^2$$

$$H_E = \frac{1}{2}\dot{q}^2 + V_E(q) = \frac{1}{2}\dot{q}^2 - \frac{\lambda}{4!}(q^2-\eta^2)^2$$

$\times$ solve for $\bar{S}_E \doteq S_E(\bar{x})$ of $S_E(q) = \int d\tau \, \frac{1}{2}\dot{q}^2 - V_E(q)$

$\bar{x}$ satisfies

$$\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} = \ddot{q} - \frac{\lambda}{4!} 2 (q^2 - \eta^2) \cdot 2q$$

$$= \ddot{q} - \frac{\lambda}{3!}(q^3 - \eta^2 q)$$

$$= \ddot{q} + \frac{\lambda \eta^2}{3!} q - \frac{\lambda}{3!} q^3 = 0$$

$$= \ddot{q} + \omega^2 q - \frac{\lambda}{3!} q^3 = 0$$

NOT EASY To solve

How about Energy conservation ?

$$\frac{1}{2}\dot{q}^2 - \frac{\lambda}{4!}(q^2 - \eta^2)^2 = 0 \quad \text{EASY!}$$

$\bar{q} = \pm \eta$ are 2- solutions

another one :

$$\frac{d\bar{q}}{d\tau} = \sqrt{\frac{\lambda}{12}(\bar{q}^2 - \eta^2)^2}$$

$$\Rightarrow \pm \int_{q_c}^{q} d\bar{q} \, (\bar{q}^2 - \eta^2)^{-1} = \sqrt{\frac{\lambda}{12}} \int_{\tau_c}^{\tau} d\tau$$

$$\mp \frac{1}{\eta} \tanh^{-1} \frac{\bar{q}}{\eta} = \sqrt{\frac{\lambda}{12}} (\tau - \tau_c) \leftarrow \text{choose } \tau_c \text{ so that when } \tau = \tau_c, \ \bar{q} \doteq 0$$

$$\Rightarrow \tanh^{-1} \frac{\bar{q}}{\eta} = \mp \sqrt{\frac{\lambda \eta^2}{12}} (\tau - \tau_c)$$

$$\bar{q} = \mp \eta \tanh\left(\frac{1}{\sqrt{2}} \omega (\tau - \tau_c)\right)$$

$$S_E(q) = \int d\tau \, \frac{1}{2}\dot{q}^2 + \frac{\lambda}{4!}(q^2 - \eta^2)^2$$

$$= \int d\tau \, \frac{1}{2}\Big[\dot{q}^2 + \frac{\lambda}{12}(q^2 - \eta^2)^2 - 2\sqrt{\frac{\lambda}{12}}(q^2 - \eta^2)\dot{q}\Big]$$

$$+ \sqrt{\frac{\lambda}{12}}(q^2 - \eta^2)\dot{q}$$

$$= \int d\tau \, \frac{1}{2}\left(\dot{q} - \sqrt{\frac{\lambda}{12}}(q^2 - \eta^2)\right)^2 + \int d\tau \, \dot{q}\,|q^2 - \eta^2| \sqrt{\frac{\lambda}{12}}$$

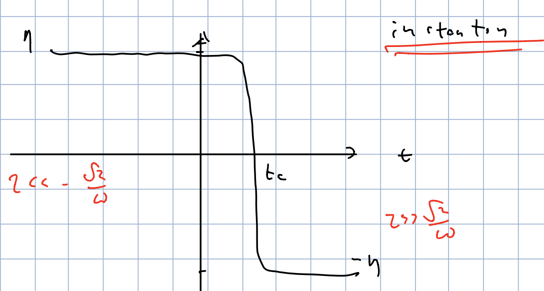
$$= \int d\tau \, \frac{1}{2}\left(\dot{q} - \sqrt{\frac{\lambda}{12}}(q^2 - q^2)\right)^2 + \int_{-\eta}^{\eta} dq \, |q^2 - \eta^2| \sqrt{\frac{\lambda}{12}}$$

$$\geq \frac{4}{3}\eta^3 \sqrt{\frac{\lambda}{12}}$$

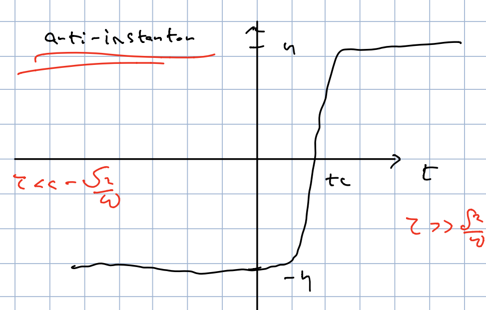
$$= \frac{4}{3} \eta \frac{\eta^2 \lambda}{\lambda} \sqrt{\frac{\lambda \eta^2}{12}}$$

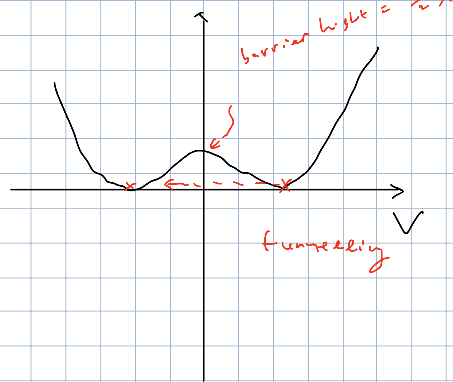
$$= 4\sqrt{2}\frac{\omega^3}{\lambda}$$

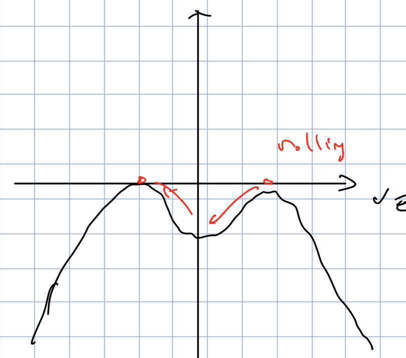
For $q = -\eta \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau - \tau_c)\right)$.



For $q = +\eta \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau - \tau_c)\right)$





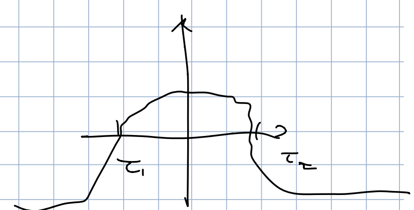


Remarks:

* Consider $\bar{x}_2 = \pm\eta \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau-\tau_1)\right) \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau_2-\tau)\right)$

$$\left.\frac{\delta S}{\delta x}\right|_{x=\bar{x}_2} = -\eta\omega^2 \cosh\left[\frac{\tau_2-\tau_1}{\sqrt{2}}\omega\right] \frac{1}{\cosh^3\left(\frac{\omega}{\sqrt{2}}(\tau-\tau_1)\right)} \frac{1}{\cosh^3\left(\frac{\omega}{\sqrt{2}}(\tau_2-\tau)\right)}$$

If $|\tau_2-\tau_1| \gg \frac{\sqrt{2}}{\omega}$



$$= -\eta\omega^2 \cosh\left[\frac{t_2-t_1}{\sqrt{2}}\omega\right] \times \cdots$$

$$= -\eta\omega^2 \exp\left[-\frac{|t_2-t_1|}{\sqrt{2}}\omega\right] \times \cdots$$

$$\longrightarrow 0$$

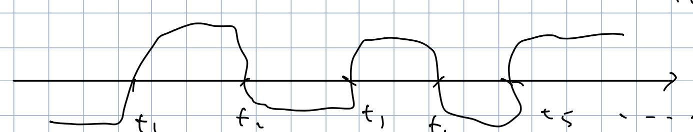
So

$\bar{x}_2$ is solution of $\frac{\delta S_E}{\delta \bar{x}} = 0$ up to extremely suppressed terms.

can be generalized to n-instantons

$$\bar{x}_n = \pm\eta \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau-\tau_1)\right) \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau_2-\tau)\right) \cdots \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau_n-\tau)\right)$$

$|t_i - t_{i+1}| \gg \frac{\sqrt{2}}{\omega}$



* Note that we have infinite solutions, since different τ_c give different solutions, but different τ_c gives exactly the same action $S_E(\bar{x})$, which is due to time-translational invariance !! So $\underline{\delta S = 0}$!

Easy to check that $\bar{q} = \mp\eta \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau-\tau_c')\right)$ gives exactly the same
and $\bar{q} = \mp\eta \tanh\left[\frac{1}{\sqrt{2}}\omega(\tau-\tau_c'')\right]$ action S_E !

AND EASY to calculate $\delta\bar{q} = \mp\eta \tanh\left(\frac{1}{\sqrt{2}}\omega(\tau-(\tau_c+d\tau_c))\right) - \left[\mp\eta\tanh\left(\frac{1}{\sqrt{2}}\omega(\tau-\tau_c)\right)\right]$

To proceed, we consider 1-instanton first. *

* 0-instanton is nothing but Harmonic oscillation $+ \frac{\lambda}{\varphi!}$ corrections

$$S_E(\bar{x}) = 0 \quad \text{for} \quad \bar{x} = \pm\eta$$

$$S_E(\bar{x}) = \int_{-T/2}^{T/2} \frac{1}{2}\dot{\bar{q}}^2 - V_E(\bar{q}) \, d\tau$$

$$= \int_{-T/2}^{T/2} \dot{\bar{q}}^2 \, d\tau = \int_{\eta}^{-\eta} \dot{\bar{q}} \, d\bar{q}$$

$$= \sqrt{\frac{\lambda}{12}} \int_{\eta}^{-\eta} (\bar{q}^2 - \eta^2) \, d\bar{q}$$

$$= \sqrt{\frac{\lambda}{12}} \frac{4}{3}\eta^3 = \sqrt{\frac{\lambda\eta^2}{12}} \frac{4\eta^2\lambda}{3\lambda} = 4\sqrt{2}\frac{\omega^3}{\lambda}$$

For both instanton & anti-instanton

See explicitly that $S_E(\bar{q})$ is Independent of τ_c !!

$$\Rightarrow \quad e^{-S_E} = e^{-4\sqrt{2}\frac{\omega^3}{\lambda}}$$

$\Rightarrow \quad \sim e^{-\bar{S}_E} \det[-\partial_\tau^2 + V''(\bar{x})]^{-1/2}$

$$= N e^{-4\sqrt{2}\frac{\omega^3}{\lambda}} \det[-\partial_\tau^2 + \omega^2]^{-1/2} \, \omega \, \frac{\det[-\partial_\tau^2 + V''(\bar{x})]^{-1/2}}{\omega \det[-\partial_\tau^2 + \omega^2]^{-1/2}}$$

for large τ, reduces to ω^{-1}

DANGEROUS!! we have zero eigenvalue here!

Why we have zero eigenvalues ??

Zero mode.

S is invariant under $\tau_c \to \tau_c + \Delta\tau_c$. (time-translation)

But $\bar{x}(\tau_c)$ does change. For different τ_c.

or to say, the time-translational invariance is broken by the vaccuum expectation !!

(the idea of spontaneous breaking)

more explicitly.

$$S[\bar{x}(\tau_c + \delta\tau_c)] - S[\bar{x}(\tau_c)] = \frac{\delta S}{\delta \bar{x}} \underbrace{\delta\bar{x}}_{\text{induced by } \tau_c \to \tau_c + \delta\tau_c} = 0$$

$$\Rightarrow \quad \frac{\delta}{\delta\bar{x}}\left(\frac{\delta S}{\delta\bar{x}}\delta\bar{x}\right) = \frac{\delta^2 S}{\delta\bar{x}\,\delta\bar{x}}\delta\bar{x} + \cancel{\frac{\delta S}{\delta\bar{x}}}\frac{\partial\delta\bar{x}}{\delta\bar{x}} = 0$$

Since $\quad \delta\bar{x} = \frac{\delta\bar{x}}{\delta\tau_c}\delta\tau_c \neq 0$

$$\Rightarrow \quad \frac{\delta^2 S}{\delta\bar{x}\,\delta\bar{x}}\delta\bar{x} = \int d\tau \left(-\partial_\tau^2 + V''(\bar{x})\right)\delta\bar{x} = 0$$

zero eigenvalue

$$\delta\bar{x} = \frac{\delta\bar{x}}{\delta\tau_c}\, d\tau_c = N \frac{1}{\sqrt{2}}\eta\omega \cosh^{-2}\left[\frac{\omega}{\sqrt{2}}(\tau-\tau_c)\right] \frac{d\tau_c}{N}$$

$$= N\frac{\delta\bar{x}}{\delta\tau_c}\left(\frac{1}{N}d\tau_c\right) \qquad \sim \Psi_0(\tau) \qquad\qquad d c_0$$

Note that $\int d\tau\, \Psi_n(\tau)\Psi_m^*(\tau) = \delta_{nm}$.

$$= N^2 \int d\tau\, \frac{\delta\bar{x}}{\delta\tau_c}\frac{\delta\bar{x}}{\delta\tau_c} = N^2\, S_E(\bar{x}) = 1$$

$$\Rightarrow \quad N = S_E^{-1/2}(\bar{x})$$

$$\Rightarrow \quad \Psi_0(\tau) = S_E^{-1/2}(\bar{x})\frac{d\bar{x}}{d\tau_c} \qquad\qquad S_E(\bar{x}) = 4\sqrt{2}\frac{\omega^3}{\lambda}$$

$$d c_0 = S_E^{1/2}(\bar{x})\, d\tau_c$$

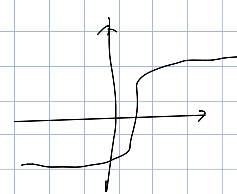
Seperate out the zero-mode

$$e^{-4\sqrt{2}\frac{\omega^3}{\lambda}}\, N \det[-\partial_\tau^2+\omega^2]^{-1/2}\, \frac{\det[-\partial_\tau^2+V''(\bar{x})]^{-1/2}_{\neq 0}}{\omega \det[-\partial_\tau^2+\omega^2]^{-1/2}} \qquad \frac{1}{\sqrt{2\pi}} S_E^{1/2}(\bar{x})\, d\tau_c\, \omega$$

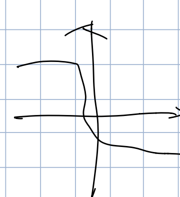
$$= \underbrace{\left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2}}_{\text{Harmonic oscillator}} \quad \underbrace{K\, e^{-S_E(\bar{x})}\, S_E^{1/2}(\bar{x})\, \frac{1}{\sqrt{2\pi}}\, d\tau_c\, \omega}_{\text{Corrections}}$$

$$N e^{-S_E(\bar{x})} \det[-\partial_\tau^2 + V''(\bar{x})]^{-1/2} = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2}\, K\, e^{-S_E(\bar{x})}\, S_E^{1/2}(\bar{x})\, d\tau_c\, \omega$$

1-instanton



or



$$\langle -\eta | e^{-HT} | \eta \rangle_4 = N \prod^4 \det[-\partial_\tau^2 + U''(\bar x)]^{-1/2} e^{-\bar S_E}$$

$$= \underbrace{N \prod^4 \det[\partial_\tau^2 + \omega^2]^{-1/2}}_{\Downarrow \text{ ** see below}} \prod^4 \frac{\det[\cdots]\ \omega}{\omega \det[-\partial_\tau^2 + \omega^2]^{-1/2}} e^{-\bar S_E}$$

$$= \underline{\underline{\left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}T}}} \omega^4 e^{-4\bar S_E} \bar S_E^{-1/2} K^4 \left(\frac{1}{2\pi}\right)^{\frac{4}{2}} \int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \int_{-T/2}^{\tau_2} d\tau_3 \int_{-T/2}^{\tau_3} d\tau_4$$

$$= \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}T} \frac{1}{4!}\left(\frac{\omega K e^{-\bar S_E}}{\sqrt{2\pi}} \bar S_E^{\frac{1}{2}}\right)^4 T^4$$

$$\Rightarrow \langle \pm\eta | e^{-HT} | \pm\eta \rangle = \sum_{n=2,4,\ldots} \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}T} \frac{1}{n!}\left(\frac{\omega K}{\sqrt{2\pi}} e^{-\bar S_E} \bar S_E^{1/2}\right)^n$$

$$= \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}T} \frac{1}{2}\left[e^{\frac{K e^{-\bar S_E}}{\sqrt{2\pi}} \bar S_E^{1/2} \omega T} + e^{-\frac{K e^{-\bar S_E}}{\sqrt{2\pi}} \bar S_E^{1/2} \omega T} \right]$$

$$\langle \mp\eta | e^{-HT} | \pm\eta \rangle = \sum_{n=1,3,5,\ldots} \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}T} \frac{1}{n!}\left(\frac{\omega K}{\sqrt{2\pi}} e^{-\bar S_E} \bar S_E^{\frac{1}{2}}\right)^n$$

$$= \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\omega}{2}T} \frac{1}{2}\left(e^{\frac{K e^{-\bar S_E}}{\sqrt{2\pi}} \bar S_E^{1/2} \omega T} - e^{-\frac{K e^{-\bar S_E}}{\sqrt{2\pi}} \bar S_E^{1/2} \omega T} \right)$$

$$\Rightarrow E_0 = \frac{\omega}{2} - \frac{K}{\sqrt{2\pi}} e^{-\bar S_E} \bar S_E^{\frac{1}{2}} \omega, \quad E_1 = \frac{\omega}{2} + \frac{K}{\sqrt{2\pi}} e^{-\bar S_E} \bar S_E^{\frac{1}{2}} \omega.$$

NON-DEGENERATE!!

**

$$\tau_0 \quad \tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \quad \tau_N$$

$$\sim \operatorname{Det}[-\partial_{\tau_1}^2 + \omega^2]^{-1/2} \operatorname{Det}[-\partial_{\tau_2}^2 + \omega^2]^{-1/2} \cdots \operatorname{Det}[-\partial_{\tau_4}^2 + \omega^2]^{-1/2}$$

$$= \mathcal{N} \int D[q(\tau_1)]\, D[q(\tau_2)]\, D[q(\tau_3)]\, D[q(\tau_4)]$$

$$\times \exp\left[- \delta q_1 \frac{\delta^2 S_E}{\delta q_1^2}\Big|_{\bar q_1} \delta q_1\right] \cdots \exp\left[- \delta q_4 \frac{\delta^2 S_E}{\delta q_4^2}\Big|_{\bar q_4} \delta q_4\right]$$

$$= \mathcal{N} \int D[q(\tau_1)]\, D[q(\tau_2)]\, D[q(\tau_3)]\, D[q(\tau_4)]$$

recall the difinition of $D(q)$

$$\times \exp\left[- \int d\tau_1 \delta q_1 \frac{\delta^2 L_E}{\delta q_1^2}\Big|_{\bar q_1} \delta q_1\right] \cdots \exp\left[- \int d\tau_4\, \delta q_4 \frac{\delta^2 L_E}{\delta q_4^2}\Big|_{\bar q_4} \delta q_4\right]$$

$$= \mathcal{N} \int D[q(\tau)] \; \exp\left[- \int_{\tau_0}^{\tau_N} d\tau \; \delta q \frac{\delta^2 L_E}{\delta q^2}\Big|_{q=\bar q} \delta q\right]$$

$$= \mathcal{N} \int D[q(\tau)] \; \exp\left[- \delta q(\tau) \frac{\delta^2 S_E}{\delta q^2}\Big|_{\bar q} \delta q(\tau)\right]$$

$$= \mathcal{N} \operatorname{Det}[-\partial_\tau^2 + \omega^2]^{-1/2}$$

$$= \left(\frac{\omega}{\pi}\right)^{1/2} \exp\left[-\frac{\omega}{2} T\right]$$

END OF "1+0" Field Theory

A.K.A Q.M.