- Feynman Rules / Feynman Diagrams

Recall that

$$|\psi, t_f\rangle_I = \left(1 + (-i)\int_{t_i}^{t_f} V_I(t)\,dt + (-i)^2 \int_{t_i}^{t_f} V_I(t)\,dt \int_{t_i}^{t} V_I(t')\,dt'\,dt' \right.$$

$$\left. + \cdots \right) |\psi, t_i\rangle_I$$

$$= \Big[1 + (-i)\int_{t_i}^{t_f} V_I(t)\,dt$$

$$+ \frac{(-i)^2}{2!} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt'\; T\left[V_I(t), V_I(t') \right] dt'\,dt$$

$$+ \cdots + \frac{(-i)^n}{n!} \int_{t_i}^{t_f} \cdots \int_{t_i}^{t_f} T\left[V_I(t_1) \cdots V_I(t_n) \right] dt_1 \cdots dt_N \Big] |\psi, t_i\rangle_I$$

$$= T\left[\exp \int_{t_i}^{t_f} V(t)\,dt \right] |\psi, t_i\rangle_I$$

Time ordering

Here

$$T\left[V_I(t_1) \cdots V_I(t_N) \right]$$

time ordering

$$= V_I(t_1) V_I(t_2) \cdots V_I(t_N)\, \theta(t_1 > t_2)\,\theta(t_2 > t_3) \cdots \theta(t_{N-1} > t_N)$$

$$+ V_I(t_2) V_I(t_1) \cdots V_I(t_N)\, \theta(t_2 > t_1)\,\theta(t_1 > t_3) \cdots \theta(t_{N-1} > t_N)$$

$+$ All permutations .

Also we note that

$$V_I = V_I(q) = V_I(a^\dagger, a)$$

So we want to find a way to simplify

$$T[V_I(a^\dagger, a), \cdots, V_I(a^\dagger, a)]$$ we ignore "I" below

without doing tedious $[a^\dagger, a]$ commutation repeatedly.

To do that, first we note that

Now, all q's are in the interacting picture, though formally they are the same as the "free" q's.

$$T[q(t_1)\, q(t_2)]$$

$$= q(t_1) q(t_2)\, \theta(t_1 - t_2) + t_1 \leftrightarrow t_2$$

$$= \frac{1}{2\omega}\left(a e^{-i\omega t_1} + a^\dagger e^{i\omega t_1}\right)$$

$$\times \left(a e^{-i\omega t_2} + a^\dagger e^{i\omega t_2}\right) \theta(t_1 - t_2) \quad + t_1 \leftrightarrow t_2$$

$$= \frac{1}{2\omega}\left[a a\, e^{-i\omega(t_1+t_2)} + \underline{a a^\dagger}\, e^{-i\omega(t_1-t_2)} + a^\dagger a\, e^{i\omega(t_1-t_2)} + a^\dagger a^\dagger e^{i\omega(t_1+t_2)}\right]$$

$$\times\ \theta(t_1 - t_2) \quad + \quad t_1 \leftrightarrow t_2$$

$$= \frac{1}{2\omega}\Big(a a\, e^{-i\omega(t_1+t_2)} + a^\dagger a^\dagger e^{i\omega(t_1+t_2)} + a^\dagger a\, e^{-i\omega(t_1-t_2)} + a^\dagger a\, e^{i\omega(t_1-t_2)}$$

$$+\ e^{-i\omega(t_1-t_2)} \Big)\, \theta(t_1 - t_2) \quad + \quad t_1 \leftrightarrow t_2$$

$$= \frac{1}{2\omega}\left[a a\, e^{-i\omega(t_1+t_2)} + a^\dagger a^\dagger e^{i\omega(t_1+t_2)} + \underline{a^\dagger a}\, e^{i\omega(t_2-t_1)} + a^\dagger a\, e^{i\omega(t_1-t_2)}\right]$$

$$+\ e^{-i\omega(t_1-t_2)} \frac{1}{2\omega}\, \theta(t_1 - t_2) + e^{-i\omega(t_2-t_1)} \frac{1}{2\omega}\, \theta(t_2 - t_1)$$

$$= \ :q(t_1) q(t_2): \ + \ \frac{1}{2\omega} e^{-i\omega|t_1 - t_2|}$$

So

$$T[q(t_1)q(t_2)] = :q(t_1)q(t_2): + \Delta_F(t_1-t_2)$$

or

$$T[q(t_1)q(t_2)] = :q(t_1)q(t_2): + \underbrace{q(t_1)q(t_2)}$$

Here we have introduced

1. Normal ordered production

$$:A(q^+,a): = a^+ \cdots a^+ a \cdots a$$

All a's on the right of all a^+'s

2. Contraction or Feynman Propagator

$$\underbrace{q(t_1)q(t_2)} = \Delta_F(t_1-t_2)$$

$$\equiv \langle 0|T[q(t_1)q(t_2)]|0\rangle = \frac{1}{2\omega}e^{-i\omega|t_1-t_2|}$$

$$\xrightarrow{t\to -i\tau} \frac{1}{2\omega}e^{-\omega|\tau_1-\tau_2|}$$

Also note that $\underbrace{q(t_1)q(t_2)} = \begin{cases} [q^+(t_1), q^-(t_2)] & t_1 > t_2 \\ [q^+(t_2), q^-(t_1)] & t_1 < t_2 \end{cases}$

$q^+(t_1) = \frac{1}{\sqrt{2\omega}} a e^{-i\omega t}$, $q^- = \frac{1}{\sqrt{2\omega}} a^+ e^{i\omega t}$

$= \frac{1}{2\omega} e^{-\omega|\tau_1-\tau_2|}$ $\longrightarrow$ Feynman Rule For propagator

Wick's Theorem :

$$T[q(t_1)\cdots q(t_n)] = :q(t_1)\cdots q(t_n): +$$

: All possible contractions :

Check : $n=1$ $\quad T[q(t_1)] = q(t_1) = :q(t_1):$

$n=2$ Correct by definition

Check . $n=3$. , Assume $t_1 > t_2$, $t_1 > t_3$

$$T[q(t_1)\, q(t_2)\, q(t_3)]$$

$$= q(t_1)\, T[q(t_2)\, q(t_3)]$$

$$= q(t_1)\, \underbrace{q(t_2)\, q(t_3)}_{} + \underbrace{q(t_1) :q(t_2)\, q(t_3):}_{①}$$

① $= \left(\underset{a}{q^+(t_1)} + \underset{a^+}{q^-(t_1)}\right) \underset{\propto\, a^+a^+\ aa}{:q(t_2)\, q(t_3):}$

Goal : to move $q^+(t_1)$ inside $: \cdots :$

$$= \underset{\propto\, a^+a^+a^+aa}{:q^-(t_1)\, q(t_2)\, q(t_3):} + q^+(t_1) :q(t_2)\, q(t_3):$$

$$= :q^-(t_1)\, q(t_2)\, q(t_3): + :[q^+(t_1), q^-(t_2)]\, q(t_3):$$
$$+ :[q^+(t_1), q^-(t_3)]\, q_2(t_2):$$
$$+ :q^+(t_1)\, q(t_2)\, q(t_3):$$

$$= :q(t_1)\, q(t_2)\, q(t_3): + :q(t_3): \underbrace{q(t_1)\, q(t_2)}_{} + :q(t_2): \underbrace{q(t_1)\, q(t_3)}_{}$$

$$\Rightarrow \quad T\left[q(t_1)\, q(t_2)\, q(t_3) \right]$$

$$= \,:q(t_1) q(t_2) q(t_3): + :q(t_3): \underbrace{q(t_1) q(t_2)} + :q(t_2): \underbrace{q(t_1) q(t_3)} + :q(t_1): \underbrace{q(t_1) q(t_2)}$$

$$= \,:\text{All possible contractions}: \quad + \quad : q(t_1)\, q(t_1)\, q(t_2) :$$

CORRECT and can be generalised to arbitary N to prove Wick's Theorem.

Sandwitch between vaccume :

$$\langle 0 | T[q(t_1) \cdots q(t_N)] | 0 \rangle = \text{All possible full contractions}$$

Example 1

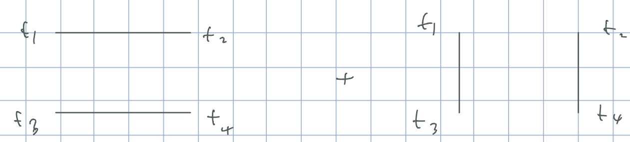
$$\langle 0 | T\{ q(t_1)\, q(t_2)\, q(t_3)\, q(t_4) \} | 0 \rangle$$

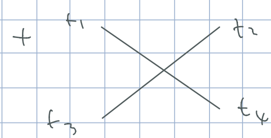
$$= \underbrace{q(t_1)\, q(t_2)}\, \underbrace{q(t_3)\, q(t_4)} + q(t_1)\, \overbrace{q(t_2)\, q(t_3)\, q(t_4)} \text{ with } \underbrace{q(t_1)\, q(t_2)\, q(t_3)}$$

$$+ \underbrace{q(t_1)\, \overbrace{q(t_2)\, q(t_3)}\, q(t_4)}$$

$$= \Delta_F(t_1 - t_2)\, \Delta_F(t_3 - t_4) + \Delta_F(t_1 - t_3)\, \Delta_F(t_2 - t_4)$$
$$+ \Delta_F(t_1 - t_4)\, \Delta_F(t_2 - t_3)$$

In terms of Feynman Diagrams





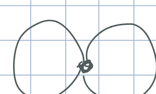
Example 2 $\qquad V[q] = \frac{\lambda}{4!} q^4$

$$-\int_0^T d\tau \, \langle 0 | T[V(q)] | 0 \rangle = \int_0^T \langle 0 | \left(-\frac{\lambda}{4!}\right) T[q(\tau)] | 0 \rangle \, d\tau$$

$$= \int_0^T d\tau \left(-\frac{\lambda}{4!}\right) \Big[q(\tau)\, q(\tau)\, q(\tau)\, q(\tau) + q(\tau)\, q(\tau)\, q(\tau)\, q(\tau)$$

$$+ \; q(\tau)\, q(\tau)\, q(\tau)\, q(\tau) \Big]$$

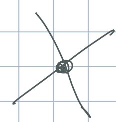
$$= \int_0^\tau d\tau$$



$$= \int_0^T d\tau \left(-\frac{\lambda}{4!}\right) \Delta_F^2(0) \times 3$$

$$= -\frac{\lambda}{4!} T \times 3 \times \left(\frac{1}{2\omega}\right)^2$$

Here we Defined

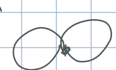
 $= \int_0^T d\tau \left(-\frac{\lambda}{4!}\right)$

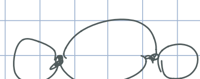
Example 3.

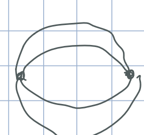
$$\frac{\left(-\frac{\lambda}{4!}\right)^2}{2!} \int_0^T d\tau_1 \int_0^T d\tau_2 \langle 0|T[V(\tau_1)V(\tau_2)]|0\rangle$$

$$= \frac{1}{2}\left(-\frac{\lambda}{4!}\right)^2 \int_0^T d\tau_1 \int_0^T d\tau_2$$

$$\times \langle 0| T[q(\tau_1)q(\tau_1)q(\tau_1)q(\tau_1)q(\tau_2)q(\tau_2)q(\tau_2)q(\tau_2)]|0\rangle$$

$= \frac{1}{2} \times \Big\{$  $\times$  $\times 9$

 $\times 6 \times 6 \times 2$

 $\times 4 \times 3 \times 2 \times 1 \Big\}$

$$
\begin{aligned}
= \quad & \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \int_0^T d\tau_1 \int_0^T d\tau_2 \Big[\Delta_F^2(0)\, \Delta_F^2(0) \times 9 \\
& \qquad + \Delta_F^2(0)\, \Delta_F^2(\tau_1-\tau_2) \times 72 \\
& \qquad + \Delta_F^4(\tau_1-\tau_2) \times 24 \Big]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{T^2}{2}\left(\frac{\lambda}{4!}\right)^2 \times 9 \times \left(\frac{1}{2\omega}\right)^4 \\
& + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \times 72 \times \left(\frac{1}{2\omega}\right)^4 \times \frac{T}{2\omega} \\
& + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \times 24 \times \left(\frac{1}{2\omega}\right)^4 \times \frac{T}{4\omega}
\end{aligned}
$$

$$
= \quad \frac{T^2}{2}\left(\frac{\lambda}{4!}\right)^2 \left(\frac{1}{2\omega}\right)^4 \times 9 + \left(\frac{\lambda}{4!}\right)^2 \left(\frac{1}{2\omega}\right)^4 \frac{21}{\omega}\, T
$$

Reproduce what we get before, but in a more efficient way?!

The equivalence between the propagator & Green function

$$\ddot{G}_F + \omega^2 G_F = \delta(t)$$

$$-k^2 \tilde{G}_F + \omega^2 \tilde{G}_F = 1 \qquad \tilde{G} = \int e^{i\omega t} G(t)$$

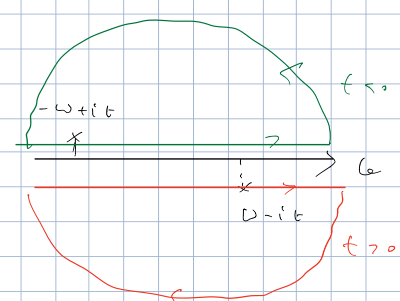
$$\tilde{G}_F = \frac{1}{\omega^2 - k^2} \sim \quad \text{Not well defined for } k = \omega$$

$$\rightarrow \quad -\frac{1}{k^2 - \omega^2 + i\epsilon}$$

to make $k=\omega$ well defined
Lots of choices, which are determined by the boundary conditions. This one corresponds to Feynman propagator.

$$G_F = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikt} \frac{1}{k^2 - \omega^2 + i\epsilon} \, dk$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikt} \frac{1}{(k - \omega + i\epsilon)(k + \omega - i\epsilon)} \, dk$$



If $t>0$, we enclose the contour in the lower semi plain.

otherwise the integral is not well-defined.

$$G_F(t>0) = \frac{1}{2\pi} 2\pi i \frac{1}{2\pi i} \oint d\omega \; e^{-i\omega t} \frac{1}{(\omega-w+i\epsilon)(\omega+w-i\epsilon)}$$

$$= \frac{1}{2\pi} 2\pi i \; e^{-iwt} \frac{1}{2w} \theta(t)$$

$$= i \, e^{-iwt} \frac{1}{2w} \theta(t)$$

If $t<0$, upper plain

$$G_F(t<0) = -\frac{1}{2\pi} 2\pi i \frac{1}{2\pi i} \oint d\omega \; e^{-i\omega t} \frac{1}{\omega-w+i\epsilon} \frac{1}{\omega+w-i\epsilon}$$

$$= -\frac{1}{2\pi} 2\pi i \; e^{iwt} \frac{1}{-2w} \theta(-t)$$

$$= i \, e^{iwt} \frac{1}{2w} \theta(-t)$$

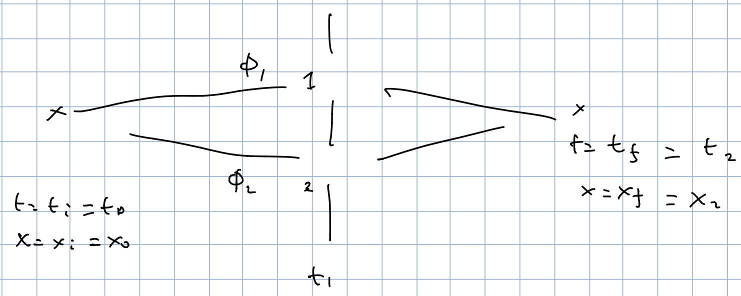
$$\Rightarrow \quad G_F(t_1-t_2) = -i \langle 0 | T(q(t_1) \, q(t_2)) | 0 \rangle$$

Feynman propagator $\propto$ Green function with special Boundary conditions.

END OF CANONICAL QUANTIZATION

Feynman Path Integral

Double slits experiment



$$P = |\phi_1 + \phi_2|^2$$

$\phi_i \equiv$ amplitude go through slit i.

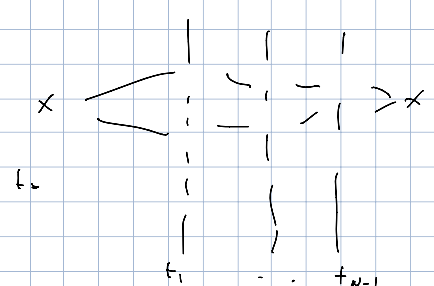
$$= \langle x_2, t_2 | x_{11}, t_1 \rangle \langle t_1\, x_{11} | x_0, t_0 \rangle$$

$$+ \langle x_2\, t_2 | x_{12}, t_1 \rangle \langle x_{12}, t_1 | x_0, t_0 \rangle$$

$$= \sum_i \phi(x_0 \to x_{1i})\, \phi(x_{1i} \to x_2)$$

more slits

$$\to \int dx_1\, \phi(x_0 \to x_1)\, \phi(x_1 \to x_2)$$



more boards

$$\to \int dx_1 \cdots dx_{N-1}$$

$$\phi(x_0 \to x_1)\, \phi(x_1 \to x_2)\, \phi(x_2 \to x_3) \cdots$$

$$\phi(x_{N-2} \to x_{N-1})\, \phi(x_{N-1} \to x_N)$$

$$\phi(x_i \to \phi_{i+1}) = \langle x_{i+1}\, t_{i+1} | x_i, t_i \rangle = \langle x_{i+1} | e^{-i\hat{H}(t_{i+1} - t_i)} | x_i \rangle$$

$$\psi = \int dx_1 \cdots dx_N \; \langle x_N | e^{-i\hat{H}(t_N - t_{N-1})} | x_{N-1} \rangle \langle x_{N-1} | e^{-i\hat{H}(t_{N-1} - t_{N-2})} | x_{N-2} \rangle \cdots \langle x_1 | e^{-i\hat{H}(t_1 - t_0)} | x_0 \rangle$$

$0 \;\; \epsilon \;\; 2\epsilon \;\; 3\epsilon \;\; \cdots \;\; (N-1)\epsilon \;\; t$ divide t to N peices

$$\langle x_{i+1} | e^{-i\hat{H}(t_{i+1} - t_i)} | x_i \rangle \qquad \hat{H} = \frac{\hat{p}^2}{2} + V(x)$$

$$= \int dp \, \langle x_{i+1} | p \rangle \langle p | e^{-i\hat{H}\epsilon} | x_i \rangle$$

$$= \frac{1}{2\pi} \int dp \; e^{ip \cdot x_{i+1}} \; e^{-i\epsilon\left(\frac{p^2}{2} + V(x_i)\right)} \; e^{-ip \cdot x_i} \qquad + O(\epsilon^2)$$

Can show it does not contribute to the final results

$$= \frac{1}{2\pi} \int dp \; e^{ip \cdot (x_{i+1} - x_i)} \; e^{-i\epsilon \frac{p^2}{2}} \; e^{-i\epsilon V(x_i)}$$

$$= \frac{1}{2\pi} \int dp \; e^{ip \cdot \dot{x}_i \epsilon} \; e^{-i\epsilon \frac{p^2}{2}} \; e^{-i\epsilon V(x_i)}$$

$$= \frac{1}{2\pi} \int dp \; e^{-i\epsilon\left(\frac{p^2}{2} - p \cdot \dot{x}_i + \frac{\dot{x}_i^2}{2}\right)} \; e^{i\epsilon \frac{\dot{x}_i^2}{2}} \; e^{-i\epsilon V(x_i)}$$

$$= \frac{1}{2\pi} \int dp \; e^{-i\frac{\epsilon}{2}(p - m\dot{x}_i)^2} \; e^{i\epsilon\left(\frac{\dot{x}_i^2}{2} - V(x_i)\right)}$$

$$= \frac{1}{2\pi} \cdot \sqrt{\frac{2\pi}{\hbar i \epsilon}} \; e^{i\epsilon L(\dot{x}, x)}$$

$$\Rightarrow \lim_{\epsilon \to 0} \left(\frac{1}{2\pi\hbar i \epsilon}\right)^{N/2} \int dx_1 \cdots dx_{N-1} \; e^{i\int dt \, L} = N \int \mathcal{D}[x] \; e^{iS}$$

$$\langle x_f t_f | x_i t_i \rangle = \lim_{\epsilon \to 0} \left(\frac{1}{2\pi i \epsilon} \right)^{N/2} \int dx_1 \cdots dx_{N-1} \; e^{iS}$$

$$= N \int D[x] \; e^{iS} \quad \longrightarrow \quad \text{highly oscillate}$$

$$\xrightarrow{t \to -i\tau} N \int D(x(\tau)) \; e^{-S_E} \quad \longrightarrow \quad \text{well defined for } S_E \geq 0$$

Time ordering intrinsically encoded in the path integral

$$L = \frac{\dot{x}^2}{2} - V(x) \xrightarrow{t \to -i\tau} -\frac{\dot{x}^2}{2} - V(x)$$

$$S \to S_E = -i \int d(-i\tau) \left[\frac{\dot{x}^2}{2} + V(x) \right] = -\int d\tau \Big(\underbrace{\frac{\dot{x}^2}{2} + V(x)}_{L_E} \Big)$$

$$\int dx \, \langle x, t_f | x, t_i \rangle = \sum_n e^{-E_n \cdot T} = \sum_n e^{-E_n \beta} \equiv Z[\beta]$$

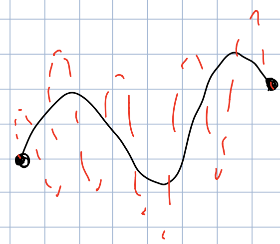
$$= \int dx \; N \int D[x_n(\tau)] \; e^{-S_E}$$

Feynman Path Integral can be used for partition function of statistics.

- Can also be used to derive the Feynman Rules. You will see this in the future.

- Here we show how this can be used for non-perturbative calculation.

General set ups



write $x(\tau)$ in terms of x_{cl}

and fluctuations

$$x(\tau) = \bar{x}(\tau) + \delta x(\tau)$$

Here

$$\left.\frac{\delta S}{\delta x}\right|_{x=\bar{x}} = 0$$

$$\Rightarrow \quad \frac{d^2\bar{x}}{d\tau^2} - V'(\bar{x}) = 0$$

$\bar{x}$ satisfies Euler - Langragian Eqn.

$$S[x] = \int \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \, d\tau$$

$$= \int \frac{1}{2}\left(\frac{d(\bar{x}+\delta x)}{d\tau}\right)^2 + V(\bar{x}+\delta x) \, d\tau$$

$$= \int \frac{1}{2}\left(\frac{d\bar{x}}{d\tau} + \frac{d\delta x}{d\tau}\right)^2 + V(\bar{x}) + V'(\bar{x}) + \frac{1}{2}V''(\bar{x})\delta^2 x$$

$$= \int \frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2 + \frac{d\bar{x}}{d\tau}\frac{d\delta x}{d\tau} + \frac{1}{2}\frac{d^2\delta x}{d\tau^2} + V(\bar{x}) + V'(\bar{x}) + \delta''(\bar{x})\delta^2 x$$

$$= \int \frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2 + V(\bar{x}) \, d\tau$$

$$+ \int \frac{dx}{d\tau}\frac{d\delta x}{d\tau} \, d\tau + \int \frac{1}{2}\left(\frac{d\,\delta x}{d\tau}\right)^2 + V'(\bar{x}) + \frac{1}{2}V''(\bar{x})\delta^2 x$$

$$= \int \frac{1}{2}\left|\frac{d\bar{x}}{d\tau}\right|^2 + V(\bar{x})\, d\tau \quad + \int \left(-\frac{d^2 x}{d\tau^2}\bigg|_{\bar{x}} + V'(\bar{x})\right)\delta x \; d\tau$$

$$\to 0$$

$$+ \int \delta x \left\{-\frac{1}{2}\frac{d^2}{d\tau^2} + \frac{1}{2}V''(\bar{x})\right] \delta x \; d\tau$$

$$= S[\bar{x}] + \int d\tau \; \delta x \left[-\frac{1}{2}\frac{d^2}{d\tau^2} + \frac{1}{2}V''(\bar{x})\right] \delta x$$

Let

$$\delta x = \sum_i c_i \, \delta x_i$$

$$\int d\tau \; \delta x_i \, \delta x_j = \delta_{ij}$$

$$\left[-\frac{d^2}{d\tau^2} + V''(\bar{x})\right) \delta x_i = \epsilon_i \, \delta x_i$$

$$\Rightarrow \quad S[x] = S[\bar{x}] + \frac{1}{2}\epsilon_i c_i^2$$

$$\Rightarrow$$

$$N \int \mathcal{D}[x]\, e^{-S[x]} = N \int \prod_i \frac{dc_i}{\sqrt{2\pi}} \; e^{-S[\bar{x}]} \; e^{\frac{1}{2}\epsilon_i c_i^2}$$

$$= e^{-S[\bar{x}]} \; N \cdot \prod_i \epsilon_i^{-\frac{1}{2}}$$

$$\text{OR} \quad \equiv e^{-S[\bar{x}]} \; N \cdot \det\left[-\frac{d^2}{d\tau^2} + V''(\bar{x})\right]^{-1/2}$$

Harmonic oscillator

if $x_i = x_f = 0 \Rightarrow \bar{x} = 0, \; S[\bar{x}] = 0$

$$\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right)\delta x_i = \epsilon_i \delta x_i$$

$$\delta x_n \propto \sin\frac{n\pi}{T}\tau \quad \text{or} \quad \cos\frac{n\pi}{T}\tau$$

$$\Rightarrow \quad \epsilon_n = \frac{n^2\pi^2}{T^2} + \omega^2 \qquad n = 0, 1, 2 \cdots$$

since $\tau = 0$ $dx = 0$
$\tau = T$ $dx = 0$

$$\Rightarrow \quad e^{-S[\bar{x}]} N \cdot \prod_n \left(\frac{n^2\pi^2}{T^2} + \omega^2\right)^{-1/2}$$

$$= e^{-S[\bar{x}]} \underbrace{N \prod_n \left(\frac{n^2\pi^2}{T^2}\right)^{-1/2}}_{\omega = 0, \text{ free particle}} \cdot \prod_n \left(1 + \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-1/2}$$

$$\langle x_f = 0 | e^{-\frac{p^2}{2}T} | x_i = 0 \rangle = \sum_n \int \frac{dp_n}{2\pi} \langle 0 | p_n \rangle \langle p_n | 0 \rangle e^{-\frac{p_n^2}{2}T}$$

$$= \sum_n \int \frac{dp_n}{2\pi} e^{-\frac{p_n^2}{2}T} = \frac{1}{\sqrt{2\pi T}}$$

$$\Rightarrow \quad = \frac{1}{\sqrt{2\pi T}} \left[\frac{1}{\omega T}\sinh \omega T\right]^{-1/2}$$

Set $S[0] = 0$ Here
for $x_i = x_f = 0$

$$= \underbrace{\left(\frac{\omega}{T}\right)^{1/2}}_{|\psi_0(0)|^2} \underbrace{e^{-\omega T/2}}_{\text{Ground energy}} \Big[1 + \underbrace{\tfrac{1}{2} e^{-\omega T}}_{n=1} + \cdots\Big]$$