Quantum Field Theory I (QFT)

- Lecturer:

  刘晓辉 week 1~8 TH

  ·吴伟 week 9-16 TH

  E-mail: xiliu@bnu.edu.cn

  OFFICE: 科技楼. C405

  OFFICE HOUR: W 2:30 - 3:30 pm

- Teaching Assistant:

  陈兆辉

  E-mail: 2017211140008@mail.bnu.edu.cn

  OFFICE: 科技楼 A309

  OFFICE HOUR: T. afternoon

- Assignments: ~8 Homework Sets due ~2 weeks

- Exam: ~40% open-book ?

- References :

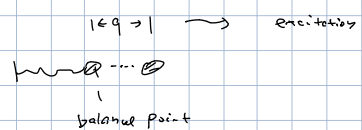
  * Srednicki , Quantum Field Theory
  * Peskin & Schroeder. An Introduction to QFT
  * Matt Shwartz , QFT & Standard Model
  * S. Weinberg , The Quantum Theory of Fields
  * David Tong , lectures on QFT
  * Wightman , PCT, spin & Statistics

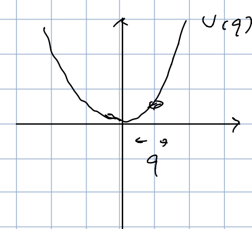
- Outline

  * motivation . why QFT ??
    - QFT $\neq$ P.T. QFT !
    - EFT.
  * Spin-0, spin-½, spin-1 Fields
  * Renormalization . (not RGE)
  * Scattering Theory, Perturbation
  * QED . Feynman Diagrams, Path Integral . .

Warm-up : A Review of QM not in a QM course

- Harmonic Oscillator





* Classic

$$L = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}\omega^2 q^2 . \qquad m = 1$$

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

$$H = p\dot{q} - L = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2$$

Equation of motion (EOM) :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \Rightarrow$$

$$\ddot{q} + \omega^2 q = 0 \qquad + \text{ B.C.}$$

$$q = a e^{-i\omega t} + a^* e^{i\omega t} \qquad + \text{ B.C.}$$

$$p = i\omega(a e^{-i\omega t} - a^* e^{i\omega t}) \qquad + \text{ B.C.}$$

DONE For Classic

* QM:

$q \to \hat{q}$, $p \to \hat{p}$ $\hbar \equiv 1$

$[\hat{q}, \hat{p}] = i$ → Canonic Quantization

$$\hat{L} = \frac{1}{2}\dot{\hat{q}}^2 - \frac{1}{2}\omega^2\hat{q}^2$$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2 , \qquad \hat{H}\psi = \underset{\substack{\uparrow \\ \text{c - number}}}{E}\psi$$

EOM:

$$\ddot{\hat{q}} + \omega^2\hat{q} = 0$$

≡ Heisenberg picture

$$\begin{cases} \hat{q} = \sqrt{\frac{1}{2\omega}}\left(e^{i\omega t}\hat{a}^{+} + e^{-i\omega t}\hat{a}\right) \\ \hat{p} = \sqrt{\frac{1}{2\omega}}\, i\omega\left(e^{i\omega t}\hat{a}^{+} - e^{-i\omega t}\hat{a}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \hat{a} = \sqrt{\frac{\omega}{2}}\left(\hat{q} - \frac{\hat{p}}{i\omega}\right)e^{-i\omega t} \\ \hat{a}^{+} = \sqrt{\frac{\omega}{2}}\left(\hat{q} + \frac{\hat{p}}{i\omega}\right)e^{i\omega t} \end{cases}$$

$$[\hat{a}, \hat{a}^{+}] = e^{-i\omega t}e^{i\omega t}\frac{\omega}{2}(i\omega)^{-1} \times \left([\hat{q}, \hat{p}] - [\hat{p}, \hat{q}]\right) = \frac{\omega}{2}(i\omega)^{-1}2i = 1$$

$$[a, a^{+}] = 1$$

$$\hat{H} = -\frac{\omega^2}{2}\left(e^{2i\omega t}\hat{a}^{+2} - \hat{a}^+ a - \hat{a}\hat{a}^+ + e^{-2i\omega t}\hat{a}^2\right)\frac{1}{2\omega}$$

$$+\frac{\omega^2}{2}\left(e^{2i\omega t}\hat{a}^{+2} + \hat{a}^+ a + \hat{a}\hat{a}^+ + e^{-2i\omega t}\hat{a}^2\right)\frac{1}{2\omega}$$

$$= \frac{\omega^2}{2}\left(\hat{a}^+ a + \hat{a}\hat{a}^+ + \hat{a}^+ a + \hat{a}\hat{a}^+\right)\frac{1}{2\omega}$$

$$= \frac{\omega^2}{2}\left(4\hat{a}^+ a + 2\right)\frac{1}{2\omega}$$

$$= \frac{\omega}{4}\left(4a^+a + 2\right) = (\hat{a}^+\hat{a} + 1/2)\,\omega$$ — reproduce what you learn in QM course

$\hat{H} = (\hat{a}^+\hat{a} + 1/2)\omega \longrightarrow$ time independent !

$[\hat{H}, \hat{a}] = [\hat{a}^+\hat{a}, \hat{a}]\omega = [\hat{a}^+\hat{a}]\hat{a}\omega = -\hat{a}\omega$

$[\hat{H}, \hat{a}^+] = a^+\omega$

Assume $|n\rangle \equiv$ eigen-state of $\hat{H}$ $\quad \hat{H}|n\rangle = E_n|n\rangle$

$\hat{H}\hat{a}|n\rangle = \hat{a}\hat{H}|n\rangle - \hat{a}\omega|n\rangle = (E_n - \omega)(\hat{a}|n\rangle)$

$\hat{H}\hat{a}^+|n\rangle = (E_n + \omega)(\hat{a}^+|n\rangle)$

$\hat{H}(\hat{a}^n|n\rangle) = (E_n - n\omega)(\hat{a}^n|n\rangle)$ . . . . . . (1)

the ground state has finite $E_0 < -\infty$

$\Rightarrow$ exist $|0\rangle$ satisfies $a|0\rangle = 0$, $\quad |0\rangle \equiv$ ground state

otherwise Eq (1) can go on forever

- Fock space

$a, \ a^+, \ |0\rangle$

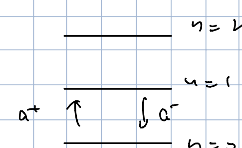
completely define the system

$$|n\rangle = \frac{a^{+n}}{\sqrt{n!}}|0\rangle \quad , \quad a|0\rangle = 0$$

$$[a, a^+] = 1 \quad , \quad \hat{H}|n\rangle = \left(n + \tfrac{1}{2}\right)\omega \, |n\rangle$$

$\hookrightarrow$ Fock Space of the Energy level

DO NOT CONFUSE THIS WITH
what later we will learn
where n is the particle number
of the Fock space there is
the Fock space of the particle numbers!!!



But nothing prevents you to take the analog.

Remember the procedure here!

You will see this again & again!

- Solve for the wave function

$\langle q | \left(\hat{q} - \frac{\hat{p}}{i\omega} \right) e^{-i\omega t} \sqrt{\frac{\omega}{2}} | 0 \rangle$ Set $\omega = 1$ for simplicity

$\Rightarrow \left(q + \frac{\partial}{\partial q} \right) \varphi_0(q) = 0$

$\Rightarrow \frac{\partial}{\partial q} \varphi_0(q) = - q \varphi_0(q)$

$\frac{\partial \varphi_0}{\varphi_0} = -q \, dq$

$\log \varphi_0 = -\frac{1}{2} q^2$

$\varphi_0 = C \cdot \exp[-\frac{1}{2} q^2]$, $\int |\varphi_0|^2 = 1 \Rightarrow C$

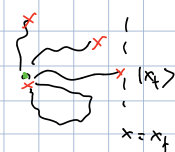
$\varphi_n(q) = \frac{\hat{a}^{+n}}{\sqrt{n!}} \varphi_0 = \frac{1}{\sqrt{n!}} \left(q - \frac{\partial}{\partial q} \right)^n \varphi_0(q)$

DONE!
EASY!
Reproduce what you learn!

- Transition amplitude

at $t=t_i$, the particle is at x_i

find the amplitude for when $t=t_f$, the particle is at X_f



• $x_i,\ t=t_i$ × $t=t_f$

$$\langle x_f | x, t=t_f\rangle \equiv \langle x_f, t_f | x_i, t_i\rangle$$

$$= \langle x_f | e^{-i\hat{H}(t_f-t_i)} | x_i\rangle$$

$$= \sum_{n,m} \langle x_f | n\rangle \langle n | e^{-i\hat{H}(t_f-t_i)} | m\rangle\langle m | x_i\rangle$$

$$= \sum_{n,m} \psi_n^*(x_f)\, \psi_m(x_i)\ e^{-iE_n(t_f-t_i)}\, \delta_{nm}$$

$$= \sum_n \psi_n^*(x_f)\, \psi_n(x_i)\ e^{-iE_n(t_f-t_i)}$$

$$\xrightarrow{x_f=x_i=x} \sum_n |\psi_n(x)|^2\ e^{-iE_n(t_f-t_i)}$$

Now we integrate over x, to find

$$\int dx\ \langle x, t_f | x, t_i\rangle = \sum_n e^{-iE_n(t_f-t_i)}$$

$$\xrightarrow[\text{Wick rotation}]{t=-iT} \sum_n e^{-E_n T} \xrightarrow[T\to\infty]{} e^{-E_0 T}$$

$$\therefore\ \lim_{T\to\infty} \frac{\log \int dx \langle x, t_f | x, t_i\rangle}{-T} = E_0$$

A different way to calculate the ground energy

A little bit more complicated.

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\lambda}{4!} q^4(t)$$

but already we do not know how to solve / Quantize, Ground energy?

- perturbation.

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle = \hat{H}_0 + \hat{V}|\psi\rangle$$

Shrodinger picture

we know how to solve $\hat{H}|\psi\rangle = E|\psi\rangle$

small

Introduce

$$|\psi\rangle_I = e^{i\hat{H}_0 t}|\psi\rangle$$

Then

$$i\frac{\partial}{\partial t}|\psi\rangle_I = -\hat{H}_0 e^{i\hat{H}_0 t}|\psi\rangle + e^{i\hat{H}_0 t} i\frac{\partial}{\partial t}|\psi\rangle$$

$$= -\hat{H}_0 e^{i\hat{H}_0 t}|\psi\rangle + e^{i\hat{H}_0 t}(\hat{H}_0 + \hat{V})|\psi\rangle$$

$$= e^{i\hat{H}_0 t} V e^{-i\hat{H}_0 t} e^{i\hat{H}_0 t}|\psi\rangle$$

$$\Rightarrow \quad i\frac{\partial}{\partial t}|\psi\rangle_I = V_I(t)\,|\psi\rangle_I$$

with

$$V_I(t) = e^{i\hat{H}_0 t} V(t) e^{-i\hat{H}_0 t}$$

$$\Rightarrow \quad |\psi\rangle_I = T\left[\exp\left[-i\int V_I(t)\,dt\right]\right]|\psi_i\rangle$$

$$= \left(1 + (-i)\int_{t_i}^{t_f} V_I(t)\,dt + (-i)^2\int_{t_i}^{t_f} V_I(t)\,dt \int_{t_i}^{t} V_I(t')\,dt' + \cdots\right)|\psi_i\rangle$$

Now we look at the transition amplitude again

$$\langle x_f | x_i\rangle = \langle x_f | e^{-i\hat{H}_0 t} | x\rangle_I \Big|_{t=t_f-t_i}$$

$$= \langle x_f | e^{-i\hat{H}_0(t_f-t_i)} \exp\left[-i\int dt\, V_I(t)\right] | x_i\rangle$$

$$= \langle x_f | n\rangle \langle n | e^{-i\hat{H}_0 t} | m\rangle \langle m | \exp\left[-i\int dt\, V_I(t)\right] | \ell\rangle \langle \ell | x_i\rangle$$

$\hat{H}_0 |n\rangle = E_n |n\rangle$ →

$$= \varphi_n^*(x_f)\, e^{-iE_n t} \langle n | \exp\left[-i\int dt\, V_I(t)\right] |\ell\rangle\, \varphi_\ell(x_i)$$

$$= \sum_{n,\ell} \varphi_n^*(x)\, \varphi_\ell(x)\, e^{-iE_n t} \langle n | \exp\left[i\int dt\, V_I(t)\right] |\ell\rangle$$

Integrate over x to get

$$\sum_n e^{-iE_n t} \langle n | \exp\left[i\int dt\, V_I(t)\right] | n\rangle$$

$$= \sum_n e^{-iE_n t} \langle n | \left(1 - i\int_0^T dt\, V_I(t) + \cdots\right) | n\rangle$$

$$\xrightarrow[T\to\infty]{t\to -i\tau,\ T\to -iT} e^{-E_0 T}\left(1 - \int_0^T d\tau \langle 0 | V_I | 0\rangle + \cdots\right)$$

$$= e^{-E_0 T}\left(1 - \int d\tau\, \langle 0 | V | 0\rangle + \cdots\right)$$

NLO corrections, (λ^1)

$$V = \frac{\lambda}{4!}\left(\sqrt{\frac{1}{2\omega}}\left(e^{\omega\tau}\hat{a}^+ + e^{-\omega\tau}\hat{a}\right)\right)^4$$

$$= \frac{\lambda}{4!}\frac{1}{4\omega^2}\left(e^{\omega\tau}\hat{a}^+ + e^{-\omega\tau}\hat{a}\right)^4$$

$$= \frac{\lambda}{4!4\omega^2}\left(e^{-\omega\tau}\hat{a}\,e^{\omega\tau}\hat{a}^+\,e^{-\omega\tau}a\,e^{\omega\tau}\hat{a}^+ + \hat{a}\hat{a}\hat{a}^+\hat{a}^+ + \cdots\right)$$

$$= \frac{\lambda}{4!}\frac{1}{4\omega^2}\left(\hat{a}\hat{a}^+\hat{a}\hat{a}^+ + \hat{a}\hat{a}\hat{a}^+\hat{a}^+\right)$$

$\hat{a}|0\rangle = |1\rangle$
$\hat{a}|1\rangle = |0\rangle$
$\hat{a}^+|0\rangle = |1\rangle$
$\hat{a}|1\rangle = |0\rangle$

$\hat{a}^+|0\rangle = |1\rangle$
$\hat{a}^+|1\rangle = \sqrt{2}|2\rangle$
$\sqrt{2}\,a|2\rangle = \sqrt{2}\sqrt{2}|1\rangle$
$2a|1\rangle = 2|0\rangle$

$$\Rightarrow \langle 0|V_I|0\rangle = \frac{\lambda}{4!}\frac{1}{4\omega^2}(1+2) = \frac{\lambda}{4!}\frac{1}{4\omega^2}\times 3 = \frac{1}{32}\frac{1}{\omega^2}$$

$$\Rightarrow \int_0^T d\tau\,\langle 0|V_I|0\rangle = \frac{\lambda}{4!}\frac{1}{4\omega^2}\cdot T$$

$$\Rightarrow \text{transition amplitude} = e^{-E_0^{(0)}T}\left(1 - \frac{\lambda}{32}\frac{T}{\omega^2}\right)$$

$$\Rightarrow E_0 = \frac{\ln\left[-E_0^{(0)}T\right] + \ln\left[1 - \frac{\lambda}{32}\frac{T}{\omega^2}\right]}{-T}$$

$$\simeq E_0^{(0)} + \frac{\lambda}{32}\frac{1}{\omega^2}$$

$$E_0 = \frac{1}{2}\omega + \frac{\lambda}{32}\frac{1}{\omega^2}$$

NLO correction

Now NNLO correction (λ^2)

$$\int_0^T d\tau \, V_I(-i\tau) \int_0^\tau d\tau' \, V_I(-i\tau')$$

already done the Wick rotation

$t \to -i\tau, \; t' \to -i\tau', \; T \to -iT$

$$= \int d\tau \int d\tau' \langle 0| e^{H_0\tau} V e^{-H_0\tau} |m\rangle\langle m| e^{H_0\tau'} V e^{-H_0\tau'} |0\rangle$$

$$= \int d\tau \int d\tau' \; \langle 0| V(-i\tau)|m\rangle\langle m| V(-i\tau)|0\rangle \, e^{-E_m(\tau-\tau')} e^{E_0(\tau-\tau')}$$

$$\langle 0|V|m\rangle = \langle 0| \frac{\lambda}{4!}\left(\sqrt{\frac{1}{2\omega}}\left(e^{\omega\tau}\hat{a}^+ + e^{-\omega\tau}\hat{a}\right)\right)^4 |m\rangle$$

$$\Rightarrow \frac{\lambda}{4!}\frac{1}{4\omega^2}\langle 0| e^{-\omega\tau} a \Big[(e^{\omega\tau}\hat{a}^+)^3 + e^{\omega\tau}\hat{a}^{+2}\hat{a} + e^{\omega\tau}\hat{a}^+\hat{a}\hat{a}^+$$

$$+ e^{\omega\tau}\hat{a}(\hat{a}^+)^2 + e^{-\omega\tau}\hat{a}\hat{a}^+\hat{a}$$

$$+ e^{-\omega\tau}\hat{a}^2\hat{a}^+ + e^{-\omega\tau}\hat{a}^+\hat{a}^2$$

$$+ (e^{-\omega\tau}\hat{a})^3 \Big] |m\rangle$$

all possible m

$$= \frac{\lambda}{4!}\frac{1}{4\omega^2} \times \Big\{$$

$$e^{-\omega\tau}\langle 1| e^{\omega\tau}\hat{a}^+\hat{a}\hat{a}^+|0\rangle + e^{-\omega\tau}\langle 1| e^{\omega\tau}\hat{a}(\hat{a}^+)^2|0\rangle$$

$$+ e^{-\omega\tau}\langle 1| e^{-\omega\tau}\hat{a}\hat{a}^+\hat{a}|2\rangle + e^{-\omega\tau}\langle 1| e^{-\omega\tau}\hat{a}^2\hat{a}^+|2\rangle + e^{-\omega\tau}\langle 1| e^{-\omega\tau}\hat{a}^+\hat{a}^2|2\rangle$$

$$+ e^{-\omega\tau}\langle 1| e^{-3\omega\tau}\hat{a}^3|4\rangle \Big\}$$

$$= \frac{\lambda}{4!}\frac{1}{6\omega^2} \times \Big\{ \left(\langle 1| \hat{a}^+\hat{a}\hat{a}^+|0\rangle + \langle 1|\hat{a}(\hat{a}^+)^2|0\rangle\right)$$

$$+ e^{-2\omega T}\left(\langle 1|\hat{a}\hat{a}^+\hat{a}|2\rangle + \langle 1|\hat{a}^2\hat{a}^+|2\rangle + \langle 1|\hat{a}^+\hat{a}^2|2\rangle\right)$$

$$+ e^{-4\omega T}\langle 1|\hat{a}^3|4\rangle \Big\}$$

$\Rightarrow$

$$\sum_m \langle 0|V|m\rangle\langle m|V|0\rangle\, e^{-E_m^{(0)}(\tau-\tau')}\, e^{E_0^{(0)}(\tau-\tau')}$$

$$= \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \times$$

$$\Big\{ \left(\langle 1|1\rangle + \langle 2|\sqrt{2}\sqrt{2}|2\rangle\right)^2$$

$$+ \left(\langle 2|\sqrt{2}\;2|2\rangle + \langle 3|\sqrt{3}\sqrt{2}\sqrt{3}|3\rangle + \langle 1|1|\sqrt{2}|1\rangle\right)^2 e^{-4\omega(\tau-\tau')}$$

$$+ \left(\langle 4|\sqrt{4}\sqrt{3}\sqrt{2}|4\rangle\right)^2 e^{-8\omega(\tau-\tau')} \Big\}$$

$$= \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \left\{ 3^2 + (6\sqrt{2})^2 e^{-4\omega(\tau-\tau')} + 4!\, e^{-8\omega(\tau-\tau')} \right\}.$$

$\Rightarrow$

$$\int_0^T d\tau \int_0^\tau d\tau' \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \left\{ 3^2 + (6\sqrt{2})^2 e^{-4\omega(\tau-\tau')} + 4!\, e^{-8\omega(\tau-\tau')} \right\}.$$

$$= \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \left\{ 3^2 \tfrac{1}{2}T^2 + (72)\frac{1}{4\omega}T + 24\left(\frac{1}{8\omega}\right)\cdot T \right\}$$

$$= \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \left(3^2 \tfrac{1}{2}T^2 + \frac{21}{\omega}\cdot T \right)$$

$$\text{transition amplitude} = e^{-E_0^{(0)}T}\left\{ 1 - \frac{\lambda}{32}\frac{T}{\omega^2} + \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \left(3^2 \tfrac{1}{2}T^2 + \frac{21}{\omega}T\right) \right\}$$

$$\Rightarrow \quad E_0 = \frac{\log[-E_0^{(0)}T) + \log[1 - \frac{\lambda}{32}\frac{T}{\omega^2} + (\frac{\lambda}{4!}\frac{1}{4\omega^2})^2 (3^2 \frac{1}{2}T^2 + \frac{21}{\omega}T))}{-T}$$

$$\simeq E_0^{(0)} + \frac{\lambda}{32}\frac{1}{\omega^2} + \cancel{\frac{1}{2}\frac{T}{\omega^4}\frac{\lambda^2}{32^2}}$$

$$- \cancel{(\frac{\lambda}{4!}\frac{1}{4\omega^2})^2 3^2 \frac{1}{2}T} - (\frac{\lambda}{4!}\frac{1}{4\omega^2})^2 \frac{21}{\omega}$$

$$E_0 = \frac{\omega}{2} + \frac{\lambda}{32}\frac{1}{\omega^2} - \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 \frac{21}{\omega}$$

DONE WITH NNLO

Comments:

- for large n, the correction is of the form

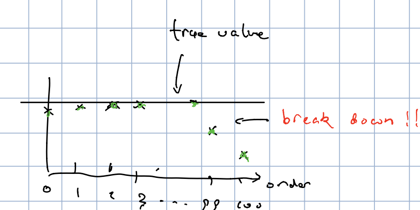
$$(-1)^{n+1}\frac{3^{n+2}}{2\pi^{3/2}}\Gamma(n+1/2)\lambda^n \sim n!\,\lambda^n$$

Bender & T.T. Wu 1969

when $n!\lambda^n \sim 1 \Rightarrow \log n! + n\log\lambda \sim 0$

$\Rightarrow n\log n + n\log\lambda \Rightarrow n \sim O(\lambda^{-1})$

P.T. starts to break down.



This means Perturbative expansion is a

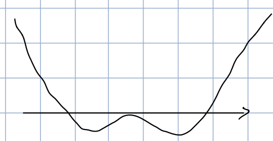
Asymptotic Series but NOT Convergent Series

How to solve? - truncate,

- Borel sum,

- nonperturbative method

- If we consider $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \frac{\lambda}{4!} q^4(t)$



perturbation can still be applied, but you will not get the true ground state

what you will get are degenerate ground states

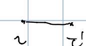
but the true ground state is NON-degenerate

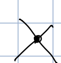
Show this in your Homework

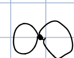
How to solve?

non-perturbative method

- Feynman Diagram / Feynman Rule

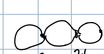
If we let:  $\equiv \frac{1}{2\omega} e^{-2\omega|\tau-\tau'|}$ called propagator

 $\equiv -\frac{\lambda}{4!}$ (after analytic continuation)

Then NLO:  $= -\frac{\lambda}{4!}\frac{1}{4\omega^2} \cdot 1 \times$ symmetry factor

$= -\frac{\lambda}{4!}\frac{1}{4\omega^2} \cdot 3$ → reproduce NLO

NNLO:

 + 

$\left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 1\, e^{-4\omega(\tau-\tau')} \times$ symmetry factor $= 72$

$+ \left(\frac{\lambda}{4!}\frac{1}{4\omega^2}\right)^2 e^{-8\omega(\tau-\tau')} \times$ symmetry factor $= 24$

 will not contribute, since it is disconnected

will see this more rigorously later.