# HW 4: Quantization of Electromagnetic field

April 28, 2019

*will be officially posted on April 18 and due on May 9 (due to the Labor Day holiday, you have one extra week).*

## 1 Problem 1: Covariant Quantization

We study how to quantize the EM field covariantly under the Lorenz gauge. We consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial \cdot A)^2\,, \tag{1}$$

which is slightly different from the Lagrangian for the free EM field. Here $\alpha$ is a dimensionless free parameter.

- Show that this Lagrangian is NOT invariant under the gauge transformation

  $$A_\mu \to A_\mu + \partial_\mu \lambda \tag{2}$$

  unless $\partial^2 \lambda = 0$. And therefore $\partial \cdot A$ is fixed to be unchanged under the gauge transformation.

- Using the Euler-Lagrangian function to show that when $\alpha = 1$, the equations of motion for the fields $A_\mu$ are

  $$\partial^2 A_\mu = 0\,. \tag{3}$$

  We note that they are exactly the same as the equations of motion for the EM fields under the Lorenz gauge.

- From now on, we work with $\alpha = 1$. Show that the canonical momenta are

  $$\pi^0 = -\partial \cdot A\,, \qquad \pi^i = \partial^i A^0 - \partial^0 A^i\,, \tag{4}$$

  in which, we can see that now $\pi^0 \neq 0$ and therefore all $A_\mu$'s are dynamical. And therefore we can propose the equal time commutation relations for all 4 components (polarizations) in this theory

  $$[A_\mu(\boldsymbol{x}), A_\mu(\boldsymbol{y})] = 0, \quad [\pi_\mu, \pi_\nu] = 0\,, \quad [A_\mu(\boldsymbol{x}), \pi_\nu(\boldsymbol{y})] = i g_{\mu\nu}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\,. \tag{5}$$

- We can expand $A_\mu$ in terms of plane wave

$$A_\mu = \sum_{r=0}^{3} \int \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\epsilon^r_\mu a_r(\boldsymbol{k}) e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + \epsilon^{r\,*}_\mu a^\dagger_r(\boldsymbol{k}) e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \equiv A^-_\mu + A^+_\mu \,, \tag{6}$$

since it satisfies the equations of motion derived before if $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$. Here we can choose

$$\epsilon^0_\mu = (1,0,0,0)\,, \quad \epsilon^1_\mu = (0,1,0,0)\,, \quad \epsilon^2_\mu = (0,0,1,0)\,, \quad \epsilon^3_\mu = (0,0,0,1)\,. \tag{7}$$

We note that the choice is not unique. We can also assume that $k$ is along the z-axis, $k = (\omega_k, 0, 0, k)$.

Show that if we propose

$$\begin{aligned} [a_r(k), a_{r'}(k')] &= [a^\dagger_r(k), a^\dagger_{r'}(k')] = 0\,, \\ [a_r(k), a^\dagger_{r'}(k')] &= \zeta_r\, \delta_{rr'} \delta^{(3)}(k - k')\,, \end{aligned} \tag{8}$$

with $\zeta_r = 1$ for $r = 1,2,3$ and $\zeta_r = -1$ for $r = 0$. The equal time commutation relation Eq. (5) can be satisfied.

- The Hamiltonian operator for this free theory can be obtained using $\hat{H} = \int \mathrm{d}^3x\, \pi^\mu \dot{A}_\mu - \mathcal{L}$, which gives

$$\hat{H} = \sum_{r=0}^{3} \int \mathrm{d}^3k \ \left(\omega_k\, \zeta_r\, a^\dagger_r(k) a_r(k) + \text{infinite vacuum energy}\right)\,. \tag{9}$$

You don't need to show this but I encourage you to do a check. We note that now all 4 polarizations contribute to the energy in this theory and the 0-component ($r = 0$, and we call this scalar photon) contributes negatively.

In order for this theory to describe correctly the EM field, we have to remove the additional 2 un-physical polarizations.

To do so, we recall that the equations of motion in this theory are equivalent to the Lorenz gauge, and therefore one way to remove the 2 unphysical degrees of freedom is to propose the Lorenz Gauge on the physical state:

$$\partial^\mu {A_\mu}^+ |\Psi_{\text{phys}}\rangle = 0\,, \qquad \langle\Psi_{\text{phys}}| \partial^\mu {A_\mu}^- = 0\,. \tag{10}$$

We emphasize that this is not the same as $\partial^\mu {A_\mu}^+ = 0$ and $\partial^\mu {A_\mu}^- = 0$ which will be satisfied for any state, physical or un-physical. The condition above is much weaker.

Now show that

$$(a_3(k) - a_0(k))|\Psi_{\text{phys}}\rangle = 0 \tag{11}$$

and therefore show that

$$\langle\Psi_{\text{phys}}|\hat{H}|\Psi_{\text{phys}}\rangle = \langle\Psi_{\text{phys}}| \sum_{r=1}^{2} \int \mathrm{d}^3k\ \omega_k \left(a^\dagger_r(k) a_r(k)\right) |\Psi_{\text{phys}}\rangle + \text{infinite vacuum energy}\,, \tag{12}$$

which reproduces the results derived in the Coulomb gauge and ONLY the transverse components are physical and contribute to the energy in the free theory.

The steps above finish the quantization of the EM field in the Lorenz gauge.

# 2 Problem 2: Lamb Shift

We try to derive Lamb shift in a more intuitive way. We already know that the E & M fields fluctuate in the vacuum which leads to a non-zero (infinite) expectation of the vacuum energy. Similar idea applies to the electron in the Coulomb potential, the fluctuation of the E & M fields will lead to a fluctuation in the electron position due to the electric force

$$m_e \frac{\mathrm{d}^2 \delta \boldsymbol{r}}{\mathrm{d}^2 t} = -e \boldsymbol{E}(\boldsymbol{x}) \,, \tag{13}$$

and in turn affects the energy level.

Here $\boldsymbol{E}(\boldsymbol{x})$ can be decomposed in terms of plane waves. For the same reason as we discussed in the class, the wave momentum $|\boldsymbol{k}| \ll m_e$ to justify the non-relativistic approximation. However different from what we have discussed in the class that the typical energy/momentum scale is $|\boldsymbol{k}| \sim m\alpha^2$, here within this approach, $\boldsymbol{k}$ has to be (much) larger than $m\alpha$ to make the electron to respond to the fluctuation of the E & M fields. Or in other words the frequency has to be higher than the typical orbiting frequency of the electron inside the hydrogen and the wavelength has to be much shorter than the Bohr radius $a_0 \sim 1/(m_e \alpha)$. Otherwise the electron will barely feel the fluctuations.

- **Solve** Eq. (13) in the Coulomb Gauge, using the fact that $\boldsymbol{E} = -\dot{\boldsymbol{A}}$ and here $\boldsymbol{A}(\boldsymbol{x}, t)$ is the quantum E & M vector field. **Write down** the displacement operator $\delta \boldsymbol{r}(\boldsymbol{x}, t)$ in terms of the creation and annihilation operators $a_r(\boldsymbol{k})$ and $a_r^\dagger(\boldsymbol{k})$ explicitly.

- The small displacement in the position will lead to a shift in the Coulomb potential

$$\begin{aligned} & \langle \phi_n^{(0)}, 0 | \Delta V | \phi_n^{(0)}, 0 \rangle \equiv \langle \phi_n^{(0)}, 0 | \left( V(\boldsymbol{x} + \delta \boldsymbol{r}) - V(\boldsymbol{x}) \right) | \phi_n^{(0)}, 0 \rangle \\ = \; & \langle \phi_n^{(0)}, 0 | \left( \delta \boldsymbol{r} \cdot \boldsymbol{\partial} V(\boldsymbol{x}) + \frac{1}{2} (\delta \boldsymbol{r} \cdot \boldsymbol{\partial})^2 V(\boldsymbol{x}) + \dots \right) | \phi_n^{(0)}, 0 \rangle \,. \end{aligned} \tag{14}$$

  Here $|\phi_n^{(0)}\rangle$ is the energy eigenstate for the Hydrogen in the Coulomb potential.

  Plug in your previous solution for $\delta \boldsymbol{r}$ and use the fact that

$$V(\boldsymbol{x}) = -\frac{Z\alpha}{|\boldsymbol{x}|} \,, \tag{15}$$

  to **find** the potential shift up to $\mathcal{O}(\alpha^2)$ and to compare with what we have done in the class for the Lamb shift.

- Note that in this case you will encounter an integral over $k$

$$\int_0^\infty \mathrm{d}k \frac{1}{k}\,, \tag{16}$$

  which is both IR and UV (logarithmic) divergent! This means that you need to specify both the UV cutoff and the IR cutoff to regulate the theory. **Try to figure out** what are the reasonable cutoff choices in this case. **Why**? And **discuss** possible solutions to eliminate the UV and IR cutoff dependences for predicting the Lamb shift.