HW 3: Relativistic QM and QFT

April 3, 2019

will be officially posted on April 4 and due on April 18.

1 Problem 1: Hydrogen Energy Level

Now we study the hydrogen energy level using Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0\,. \tag{1}$$

With the presence of the stationary Coulomb potential, we have the replacement (canonical momentum)

$$i\partial_t \to i\partial_t - e\Phi\,, \qquad -i\boldsymbol{\partial} \to -i\boldsymbol{\partial} - e\mathbf{A} \tag{2}$$

with

$$\Phi = -\frac{1}{4\pi}\frac{e}{r}\,, \qquad \mathbf{A} = 0\,. \tag{3}$$

Hence in the Coulomb potential the Dirac equation is given by

$$\left[\left(i\partial_t + \frac{\alpha}{r}\right)\gamma_0 + i\boldsymbol{\gamma}\cdot\boldsymbol{\partial} - m\right]\psi = 0\,, \tag{4}$$

with $\alpha = \frac{e^2}{4\pi}$.

- **Show that** the Dirac equation can be modified to

$$\left[\left(i\partial_t + \frac{\alpha}{r}\right)^2 + \boldsymbol{\partial}^2 + i\begin{pmatrix}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\end{pmatrix}\frac{\alpha}{r^2} - m^2\right]_{4\times 4}\psi = 0 \tag{5}$$

- **Show that** by proposing the solutions

$$\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} e^{-iEt} \tag{6}$$

and the spherical coordinate (r,θ,ϕ), the Dirac equation can be further modified to

$$\left[\frac{1}{2E}\left(-\partial_r^2+\frac{\hat{L}^2-\alpha^2\mp i\alpha\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}}{r^2}\right)-\frac{\alpha}{r}\right]_{2\times 2} r\psi_\pm=\frac{E^2-m^2}{2E}\,r\,\psi_\pm \tag{7}$$

with $\hat{\boldsymbol{r}}=\boldsymbol{r}/r$ is a unit vector. Note that the only difference from the Klein-Gordon equation is the additional spin term $\pm i\alpha\frac{\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}}{r^2}$.

- To solve for the energy level, we choose the eigenstates for total angular momentum $\boldsymbol{\mathcal{J}}=\boldsymbol{L}+\frac{1}{2}\boldsymbol{\sigma}$:

$$\psi_\pm=\frac{h_\pm(r)}{r}\,\mathcal{Y}_{jm}\left(j-\frac{1}{2},\frac{1}{2}\right)+\frac{g_\pm(r)}{r}\,\mathcal{Y}_{jm}\left(j+\frac{1}{2},\frac{1}{2}\right) \tag{8}$$

with

$$\mathcal{Y}_{jm}\left(j-\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix}\sqrt{\frac{j+m}{2j}}Y_{j-\frac{1}{2},m-\frac{1}{2}}\\ \sqrt{\frac{j-m}{2j}}Y_{j-\frac{1}{2},m+\frac{1}{2}}\end{pmatrix}\,,\quad \mathcal{Y}_{jm}\left(j+\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix}-\sqrt{\frac{j-m+1}{2j+2}}Y_{j+\frac{1}{2},m-\frac{1}{2}}\\ \sqrt{\frac{j+m+1}{2j+2}}Y_{j+\frac{1}{2},m+\frac{1}{2}}\end{pmatrix}\,, \tag{9}$$

where $\hat{L}^2Y_{l,m}=l(l+1)Y_{l,m}$ and $\mathcal{J}^2\mathcal{Y}_{jm}=j(j+1)\mathcal{Y}_{jm}$.

Show that (if it is too hard, just move on to the next question)

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\,\mathcal{Y}_{j\pm\frac{1}{2},\frac{1}{2}}=-\mathcal{Y}_{j\mp\frac{1}{2},\frac{1}{2}}\,. \tag{10}$$

- Hence the Dirac equation can be modified to

$$\left[\frac{1}{2E}\left(-\partial_r^2+\frac{1}{r^2}B_\pm\right)-\frac{\alpha}{r}\right]\begin{pmatrix}h_\pm\\ g_\pm\end{pmatrix}=\frac{E^2-m^2}{2E}\begin{pmatrix}h_\pm\\ g_\pm\end{pmatrix}\,, \tag{11}$$

where

$$B_\pm=\begin{pmatrix}(j-\frac{1}{2})(j+\frac{1}{2})-\alpha^2 & \pm i\alpha\\ \pm i\alpha & (j+\frac{1}{2})(j+\frac{3}{2})-\alpha^2\end{pmatrix} \tag{12}$$

Show that the eigenvalues of both matrices $B_\pm$ are identical and can be written as $\lambda_1(\lambda_1+1)$ and $\lambda_2(\lambda_2+1)$ with

$$\lambda_1=\sqrt{(j+1/2)^2-\alpha^2}\,,\qquad \lambda_2=\sqrt{(j+1/2)^2-\alpha^2}-1\,. \tag{13}$$

and therefore the diagonalized equations take the same form as the Klein-Gordon equation.

- **Show that** the energy levels predicted by the Dirac equation are

$$E_{\lambda_1} = m\left(1+\frac{\alpha^2}{\left(n'+\sqrt{(j+\frac{1}{2})^2-\alpha^2}\right)^2}\right)^{-1/2}, \quad \text{with } l = j-\frac{1}{2}$$

$$E_{\lambda_2} = m\left(1+\frac{\alpha^2}{\left(n'-1+\sqrt{(j+\frac{1}{2})^2-\alpha^2}\right)^2}\right)^{-1/2}, \quad \text{with } l = j+\frac{1}{2}, \quad (14)$$

where

$$n' = 1, 2, \dots . \quad (15)$$

If we express the energy level using the principle number n from the Shrodinger equation, we have (to see this we note that as $\alpha \to 0$, we need the energy level to reproduce $E_n = -\frac{1}{2}m\frac{\alpha^2}{n^2}$)

$$E_{\lambda_1} = m\left(1+\frac{\alpha^2}{\left(n-j-\frac{1}{2}+\sqrt{(j+\frac{1}{2})^2-\alpha^2}\right)^2}\right)^{-1/2}, \quad \text{with } l = j-\frac{1}{2}$$

$$E_{\lambda_2} = m\left(1+\frac{\alpha^2}{\left(n-j-\frac{1}{2}+\sqrt{(j+\frac{1}{2})^2-\alpha^2}\right)^2}\right)^{-1/2}, \quad \text{with } l = j+\frac{1}{2}, \quad (16)$$

We see that E_{λ_1} and E_{λ_2} are degenerate.

Think, what happens to the energy if α is very large? This happens to the case in which the atom number $Z \gg 1$, and $\alpha \to \alpha Z \gg 1$ for large Z. Note that for ground state, usually $j+1/2 \sim \mathcal{O}(1)$.

2 Problem 2: Gauge Redundancy and Interaction

In this problem, we study the constraints on the interaction between the EM field and the Dirac field due to the gauge redundancy. The free Lagrangian (density) for the EM field and the Dirac field is given by

$$\mathcal{L}_{\text{free}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi\,, \quad (17)$$

- First let us study the dimensions of these fields A_μ and ψ. In the natural units we introduced in class, the action is dimensionless $[S] = 0$. Show that

$$[\mathcal{L}] = 4\,, \quad [F_{\mu\nu}] = 2\,, \quad [A_\mu] = 1\,, \quad [\psi] = \frac{3}{2}\,.$$

- Show explicitly that the free Lagrangian $\mathcal{L}_{\text{free}}$ is NOT invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)\,, \quad \psi(x) \to \psi e^{i\lambda(x)}\,, \quad \bar{\psi}(x) e^{-i\lambda(x)}\,. \tag{18}$$

- Show that the if we add an additional interaction term

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \bar{\psi}\gamma^\mu \psi A_\mu\,, \tag{19}$$

the new Lagrangian is invariant under the gauge transformation.

We note that to satisfy Lorentz invariance, $\mathcal{O}_4 = \bar{\psi}\gamma^\mu\psi A_\mu$ is the interaction term with the LOWEST dimension ($[\mathcal{O}_4] = 4$) which can be constructed involving both ψ and A_μ (or F_μ) fields. One can definitely construct more interaction terms which satisfy both Lorentz invariance and gauge invariance, for instance, $\mathcal{O}_7 = \bar{\psi}\psi F^{\mu\nu}F_{\mu\nu}$. You can check that this interaction term is itself invariant under the gauge transformation above. But its unit is $[\mathcal{O}_7] = 7$. Therefore in order to include this term in the Lagrangian, it has to have the form $\frac{1}{\Lambda^3}\bar{\psi}\psi F^{\mu\nu}F_{\mu\nu}$, with $[\Lambda] = 1$ to make the dimension correct (recall that $[\mathcal{L}] = 4$). Here Λ is some energy scale and for some reason called *naturalness* Λ is usually taken to be very large (around the energy scale where the field theory starts to break down), much larger than the typical energy scale (the energy/momentum of the EM field or the Dirac field) in the problem. Immediately you can see that the effects of $\mathcal{O}_7$ will be highly suppressed by Λ^3 compare to $\mathcal{O}_4$ and thus negligible as $\Lambda \to \infty$.