Introduction to Transverse Momentum Dependent Parton Distributions (TMDs)

reference: TMD Handbook
arXiv: 2304.03302

plan of the Lecture

- TMD Factorization
- gauge link
- polarized TMDs

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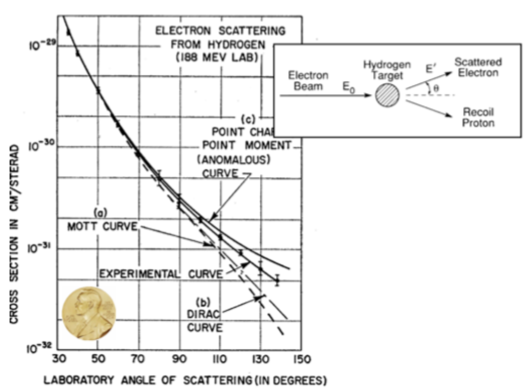
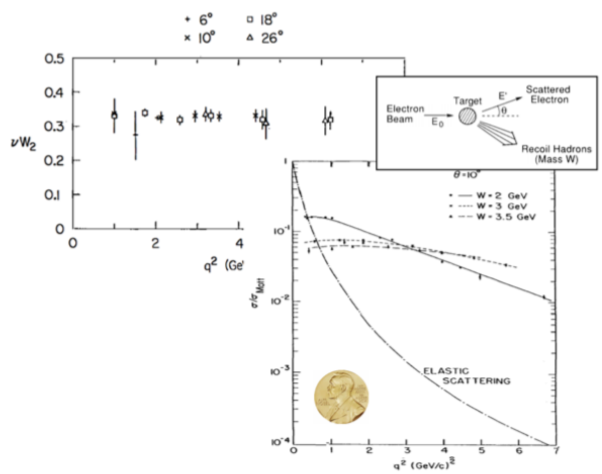
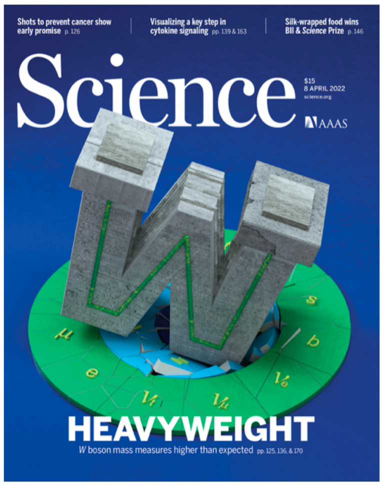
Motivation - Why hadron structures?

- a Subject with a glory history
  - proton is NOT fundamental 1950's
  - Deep Inelastic Scattering 1960's – 1970's
    + Parton model
    + Asymptotic freedom
    + Discovery of QCD





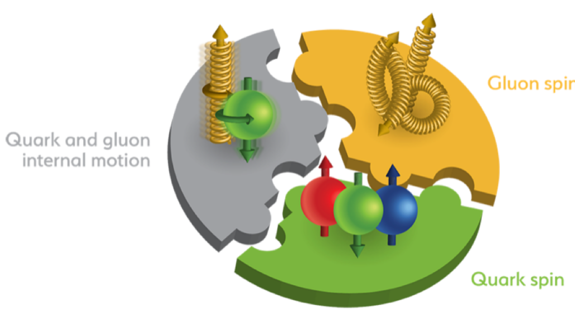
- Crucial Inputs to the LHC physics



- Major uncertainty

  e.g. Higgs, $W^{+/-}$ ...

- Playground for Studying Non-Perturbative QCD



- Proton Spin
- Origin of mass
- gluon Saturation

  ⋮

Major Focus of

the Future EIC & EicC

What are TMDs?

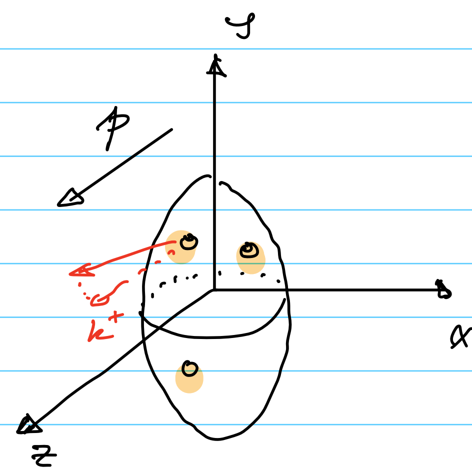
- Collinear PDFs

$$f_{i/p}(\xi, \mu), \qquad \xi = \frac{k^+}{p^+}, \qquad p^\pm \equiv p^0 \pm p^3$$

$\simeq$ probability to find a parton i with longitudinal momentum fraction ξ.

See Jun Gao's lecture

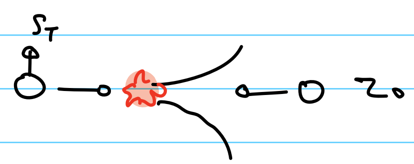
- the best known PDFs
- Can be polarized, L/L + T/T
- The information is incomplete.

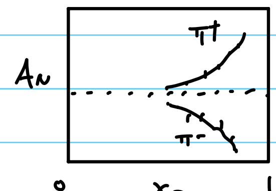


+ transverse d.o.f's are missing,
  e.g. momentum?
  Polarization structures?

+ Observed Single Spin Asymmetry larger than the (naive) collinear expectation $\propto \frac{4\alpha_s}{3}\frac{m_q}{Q} \to 0$

$\lesssim 60\%$





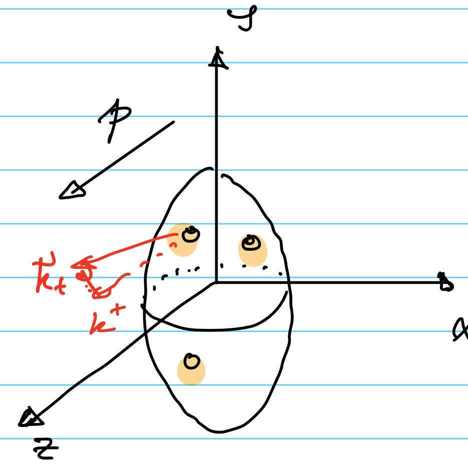
Consistently observed for 40 years!!

- TMD PDFs

$f(\xi, \vec{k}_t, \mu, \upsilon)$

υ → rapidity scale

$\vec{k}_t$ → transeverse $\vec{k}_t = (k_x, k_y)$



≃ Probability to find a parton i with longitudinal momentum fraction ξ and transeverse momentum k_t

8 TMD PDFs unpolarized + polarized

		Parton Polarization		
		unpol.	longitudinal	transverse
Proton Polarization	U	$f_1 =$		$h_1^\perp =$ − Boer-Mulders
	L		$g_1 =$ − helicity	$h_{1L}^\perp =$ − worm-gear
	T	$f_{1T}^\perp =$ − Sivers	$g_{1T}^\perp =$ − Worm-gear	$h_1 =$ − transversity $h_{1T}^\perp =$ − Pretzelosity

FF

	U	T	
U	$D_1 =$	$H_1^\perp =$ −	Collins

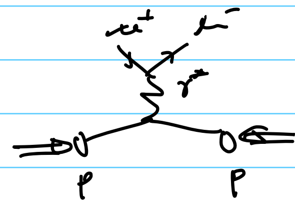
will explain later

Some remarks

– How to probe ?

- Smash the proton

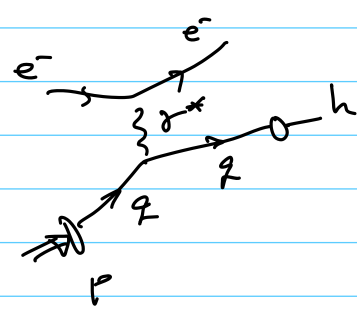
+ Drell-Yan : $pp \to \gamma^* \to \mu^+\mu^-$



$\rightsquigarrow$ $Q^2 \gg |\vec{q}_t| \sim \Lambda_{QCD}$ of the $\mu^+\mu^-$ system

c.f. $Q^2 \sim |\vec{q}_t| \gg \Lambda_{QCD}$ for collinear PDFs.

+ Semi-inclusive DIS (SIDIS) : $\ell + p \to \ell + h(\vec{q}_t) + X$



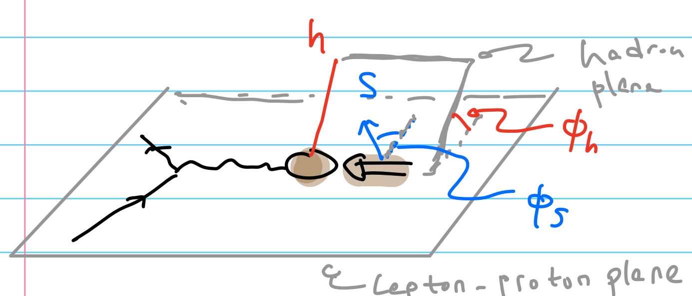
$\rightsquigarrow$ $Q^2 \gg |\vec{q}_t| \sim \Lambda_{QCD}$ of the tagged hadron h

c.f. Q^2 for collinear PDFs.

+ the beams can be polarized

e.g. $\ell(\ell,\lambda) + P(p,S) \to \ell(\ell') + h(q_t) + X$

helicity (λ), spin (S), ← unpol. (h)



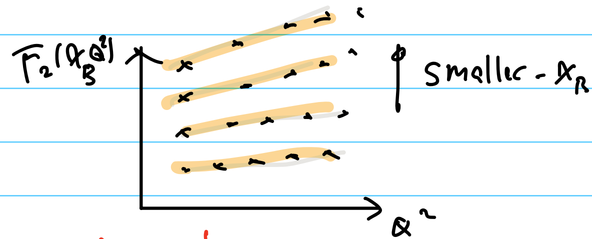
$$d\sigma \sim F_{UU,T} + \underbrace{\cos(2\phi_h)\, p_1\, F_{UU}^{\cos(2\phi_h)}}_{\text{Boer-Mulders}}$$

$$+ S_T \underbrace{\sin(\phi_h - \phi_s)\, F_{UT}^{\sin(\phi_h-\phi_s)}}_{\text{Sivers}}$$

$$+ \cdots$$

- global fitting via the factorization theorem

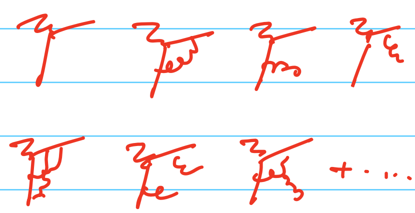
Collinear PDF:



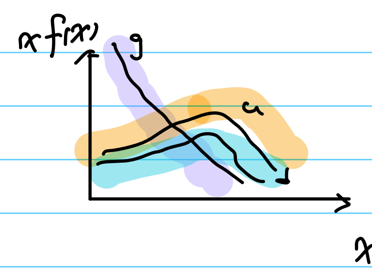
data input

$$\sigma = \hat{\sigma}_i(\mu) \otimes f_{i/p}(x_B, \mu)$$

pQCD



global fitting



Similarly for TMDs with 2-D data (x, k_t)

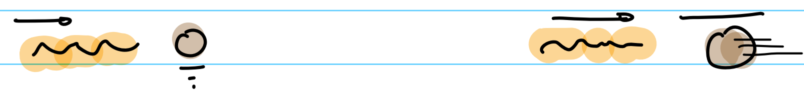
+ TMD factorization

— which Proton do we probe?

- The proton that satifies the factorization Theorem

  i.e. the proton is "infinitely" boosted

fixed-target v.s. collider



σ (data) is boost invariant

extracted PDF for the inf. boosted Proton!

TMD Factorization

let's start with reviewing the collinear factorization

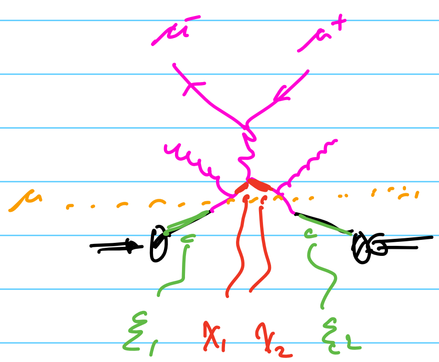
suppose we measure Q^2 & Y in Drell-Yan. $Q \gg \Lambda_{QCD}$

$$\frac{d^2\sigma}{dQ^2 dY} \sim \int_{x_1}^{1} d\xi_1 \int_{x_2}^{1} d\xi_2 \, \hat{H}_{ij}\left(\frac{x_1}{\xi_1}, \frac{x_2}{\xi_2}, \mu\right) f_{i/p}(\xi_1, \mu) \, f_{j/p}(\xi_2, \mu)$$

μ: scale to separate hard & collinear

$\hat{H}_{ij} \simeq \frac{d^2\hat{\sigma}_{ij}}{dQ^2 dY}$ Partonic x-sec : $p_i p_j \to \mu^+\mu^-$

$x_1 = \frac{Q}{\sqrt{s}} e^{Y}$. $x_2 = \frac{Q}{\sqrt{s}} e^{-Y}$. $\boxed{x_1 x_2 s = Q^2}$



hard process : $p_1 p_2 \to \mu^+\mu^- + X$

momentum $\sim Q > \mu$

a hard emission is allowed in this case

μ to seperate the scales,

long distance collinear physics

transverse momentum $< \mu$

See SCET lecture by Dingyu.

$\Rightarrow$ $\xi_i > x_i$: the convolution structure

$\Rightarrow$ $\frac{d\sigma}{d\ln\mu} = 0$ scale independent

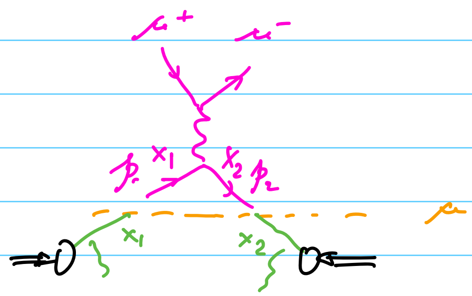
$$\Rightarrow \frac{dH}{d\ln\mu} = -\sum_i \frac{df_i}{d\ln\mu}$$

Now suppose additionally we measure $\vec{q}_t$ of the $\mu^+\mu^-$ pair.

if $|\vec{q}_t| \sim Q$ the collinear fact. still holds.

if $|\vec{q}_t| \ll Q$. the collinear fact. breaks down due to

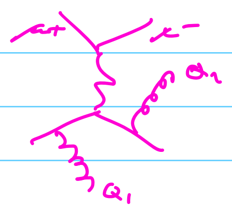
large logs of $\frac{q_t}{Q}$. $\log\frac{q_t}{Q}$. we need a TMD fact.



hard process $p_1 p_2 \to \mu^+\mu^- + \not{\phi}$ nothing,

hard radiation w/ momentum $\sim Q > \Lambda$

is power suppressed!



$Q_1 + Q_2 = q_t \ll Q$

the phase-space is so limited. the chance is low!!

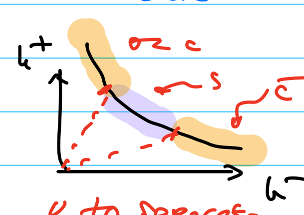
$\Rightarrow$ essentially only loops in the hard H_{ij}

long distance: both collinear & soft radiations. $\lambda \equiv \frac{|q_t|}{Q}$

$$k_c = Q(\overset{+}{1}, \overset{\perp}{\lambda}, \overset{-}{\lambda^2}) \quad k_{\bar{c}} = Q(\overset{-}{1}, \overset{\perp}{\lambda}, \overset{+}{\lambda^2}) \quad k_s = Q(\lambda, \lambda, \lambda)$$

forward backward central

with



$$\vec{k}_{t,c} + \vec{k}_{t,\bar{c}} + \vec{k}_{t,s} = \vec{q}_t$$

ν to separate the regions.

$\Rightarrow$ a rapidity scale ν to define c & S.

$$\frac{d^5\sigma}{dQ^2 dY d^2\vec{q}_t} = \hat{H}_{ij}(x_1 x_2, \mu)$$

$$\int d\vec{k}_{ct}\, d\vec{k}_{\bar{c}t}\, d\vec{k}_{s,t}\, \delta^{(2)}(\vec{k}_{ct} + \vec{k}_{\bar{c}t} + \vec{k}_{s,t} - \vec{q}_t)$$

$$\times\ \tilde{f}_{i/p}(x_1, \vec{k}_{ct}, \mu, \nu) \times \tilde{f}_{j/p}(x_2, \vec{k}_{\bar{c}t}, \mu, \nu)\ \tilde{S}(\vec{k}_{s,t}, \mu, \nu)$$

$$\frac{d\sigma}{d\ln\mu} = 0\ , \quad \frac{d\sigma}{d\ln\nu} = 0$$

TMD factorization

$\vec{b}$ space :

$$\frac{d^5\sigma}{dQ^2 dY d^2\vec{q}_t} = \hat{H}_{ij}(x_1 x_2, \mu)$$

$$\int d^2\vec{b}_t\, e^{i\vec{q}_t\cdot\vec{b}_t}\, \tilde{f}_{i/p}(x_1, \vec{b}, \mu, \nu)\, \hat{f}_{j/p}(x_2, \vec{b}, \mu, \nu)\, \hat{S}(\vec{b}, \mu, \nu)$$

$$= \hat{H}_{ij}(x_1 x_2 \mu) \int d^2\vec{b}_t\, e^{i\vec{q}_t\cdot\vec{b}_t}\, f_{i/p}(x_1, \vec{b}, \mu)\, f_{j/p}(x_2, \vec{b}, \mu)$$

where

$$f_{i/p} = \tilde{f}_{i/p} \cdot \sqrt{\tilde{S}}$$

ν cutoff cancels here

can be generalized to polarized cases.

Definition of the TMDs & the gauge link

- the TMD definition

$$f(\xi, k_t) = \int db\, e^{-ib\cdot k} \langle P | \bar{\psi}_\alpha(b^-) \frac{\gamma^+_{\alpha\beta}}{2} W^+[\gamma'] W[\gamma] \psi_\beta(0) | P \rangle$$

$$b^\mu = (\overset{+}{0}, \overset{\perp}{b_t}, \overset{-}{b^-}) \quad , \quad b\cdot k = \tfrac{1}{2} b^- k^+ - b_t \cdot k_t .$$

ψ is the collinear quark field. "good component" of the quark

$\gamma^+ = \gamma_0 + \gamma_3 = \sqrt{2}\,\bar{n}$. necessary for the coll. quark spin structure (un-polarized here) See SCET lecture

$W(\gamma)$ is the gauge link. Process dependent

⇒ gauge invariance

⇒ Originated from the inf. boost Picture

⇒ final / initial state interaction

⇒ Factorization breaking effect

The most general form:

$$\Phi_{\beta\alpha}(P, k, S) = \int db\, e^{-ib\cdot k} \langle PS | \bar{\psi}_\alpha(b^-) W^+ W \psi_\beta(0) | PS \rangle$$

Proton spin

quark spin

Will be back to this later.

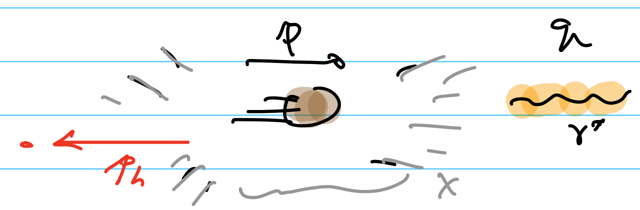
— gauge Link

We use the (SI)DIS process to highlight how the Wilson line emerges.

DIS $\quad Q^2 = -q^2$ the photon virtuality

$$x_B = \frac{Q^2}{2p\cdot q} \quad \text{Bjorken - } x\ ,$$

Breit · frame



$$q^\mu = (0,\ 0,\ 0,\ -Q)$$

$$p^\mu = \frac{Q}{2x_B}(1,\ 0,\ 0,\ 1)$$

$$p_h^\mu \simeq x_B p^\mu + q^\mu = \frac{Q}{2}(1,0,0,-1)$$

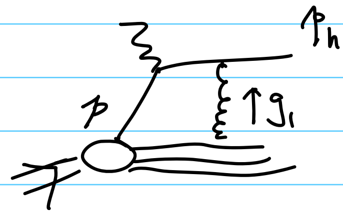
(inf boost along the proton

$Q \longrightarrow \infty$. with x_B fixed

- X compressed into a line $\bar{n}$
- a color source of gluons. propagating along n !

$\Longrightarrow$ Wilson line

lets consider one emission in SIDIS



$\boxed{p_h^- \to \infty}$.

$g_1 // p \Rightarrow g_1^+ >> \boxed{g_1^- \to 0}$

We assume $p_h^\mu = p_h^- \frac{\bar{n}^\mu}{2}$ & $\not{p}_h u_h = 0 = \underline{\underline{\bar{\not{n}} u_h}}$

$$\text{diag} = i g_s \bar{u}_h \gamma^\alpha \, i \frac{\not{p}_h - \not{g}_1}{(p_h - g_1)^2 + i0} \cdots \langle X | A_\alpha(g_1) \psi | P \rangle \cdots$$

$$\gamma^\alpha = \cancel{\frac{\bar{\not{n}}}{2} n^\alpha} + \frac{\not{n}}{2} \bar{n}^\alpha + \gamma_t^\alpha \ . \quad (p_h - g_1)^2 \simeq - p_h^- g_1^+ - g_t^2 \equiv D$$

0

$$\text{diag} = i g_s \bar{u}_h \frac{\not{n}}{2} \, i \frac{\not{p}_h - \not{g}_1}{D + i0} \cdots \langle X | \bar{n} \cdot A(g_1) \psi | P \rangle \cdots \quad ①$$

$$+ \, i g_s \bar{u}_h \gamma_t^\alpha \, i \frac{\not{p}_h - \not{g}_1}{D + i0} \cdots \langle X | A_\alpha(g_1) \psi | P \rangle \cdots \quad ②$$

① : $\not{q}_1 = \cancel{q_1^+ \frac{\not{n}}{2}}^{\to 0} + \underbrace{q_1^- \frac{\bar{\not{n}}}{2} + \not{q}_{1t}}_{\text{small component}} \Rightarrow \not{p}_h - \not{q}_1 \simeq p_1^- \frac{\bar{\not{n}}}{2}$

$\not{p}_h = p_1^- \bar{\not{n}}/2$

$$① = i g_s \bar{u}_h \frac{\not{n}}{2}\, i\, \frac{\overbrace{p_1^- \bar{\not{n}}/2}^{=1}}{-p_h^- q_1^+ - \vec{q}_t^{\,2} + i0} \cdots \langle X | \bar{n}\cdot A(q_1)\, \psi | P\rangle$$

$$= g_s \bar{u}_h \frac{1}{q_1^+ + \cancel{\frac{q_t^2}{p_h^-}}^{\to 0} - i0} \cdots \langle X | \bar{n}\cdot \overbrace{A(q_1)\, \psi | P\rangle}^{q_1 + p = P} \qquad p_h^- \to \infty$$

$$① = \bar{u}_h g_s \frac{1}{\underset{=}{q_1^+} - i0} \langle X | \bar{n}\cdot A(q_1)\, \psi | P\rangle$$

eikonal approx. $\Rightarrow$ Wilson lines.

out-going eikonal line ;

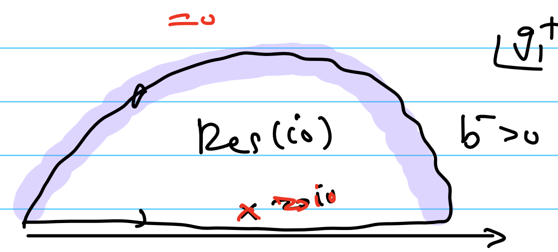
Now we take the Fourier transformation to the position space

by noting that

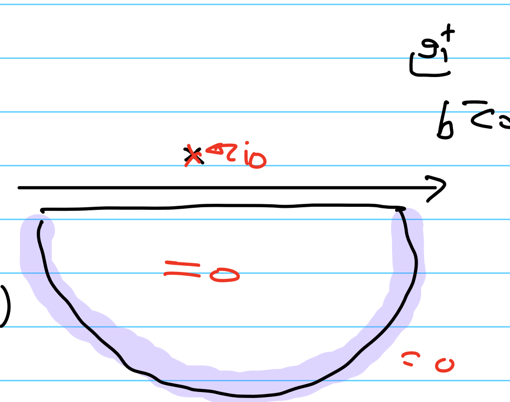
$$i \int \frac{dq_1^+}{2\pi i} \frac{1}{q_1^+ - i0}\, e^{\underset{\text{\textasciitilde}}{i q_1^+ b^-/2}}$$

$\sim i\, i\, \mathrm{Im}(q_1^+)\, b^-$

$\sim -\mathrm{Im}(q_1^+)\, b^-$



$= i\,\theta(b^-)$



$$① = \cdots i g_s \int^{+\infty} \frac{db_1^-}{2} \bar{n}\cdot A(b_1^-, \vec{b}_t)\, \theta(b_1^- - b^-)\, \psi(b^-, \vec{b}_t)$$

$$= \cdots i g_s \int_{b^-}^{+\infty} db_1^\mu A_\mu(b_1^-, \vec{b}_t)\, \psi(b^-, \vec{b}_t)$$ Standard Wilson line

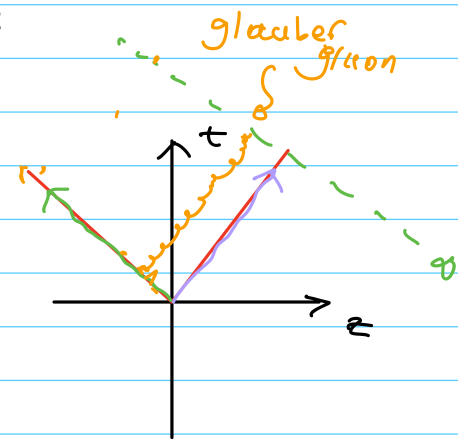
see SCET Lecture.

$$② \quad + \, i g_s \bar{u}_n \underbrace{\gamma_t^\alpha}_{\text{transverse}} \, i \, \frac{\cancel{p_n} - g_1}{D + i0} \cdots \langle X | A_\alpha(g_1) \psi | P \rangle$$

$\to 0$

② vanishes as $p_n^- \to \infty$, unless $g_1^+ \to 0$ faster.

this typically happens when the gluon is exchanged between particles at $x^- \to \infty$ in this case $g^+ \to 0$



$$\Rightarrow ② = i g_s \bar{u}_n \gamma_t^\alpha \, i \, \frac{-g_t}{-\vec{g}_t^{\,2} + i0} \cdots A_\alpha(g_1) \psi | P \rangle$$

$$= i g_s \bar{u}_n \gamma_t^\alpha \, i \, \frac{g_t}{\vec{g}_t^{\,2} - i0} \cdots A_\alpha(g_1) \psi | P \rangle$$

again we use fourier transformation

$$\int \frac{d\vec{g}}{(2\pi)^2} \frac{\vec{g}}{\vec{g}^{\,2} - i0} e^{-i \vec{g} \cdot \vec{b}_t} = i \vec{\partial} \int \frac{d\vec{g}}{(2\pi)^2} \frac{1}{\vec{g}^{\,2} - i0} e^{-i \vec{g} \cdot \vec{b}_t}$$

$$= -\frac{i}{2\pi} \vec{\partial} \ln |\vec{b}_t| = \frac{-i}{2\pi} \frac{\vec{b}_t}{\vec{b}_t^{\,2}}$$

$$\Rightarrow ② = \cdots \frac{i g_s}{2\pi} \int d\vec{b}_{1t} \vec{A}_t(\infty, \vec{b}_{1t}) \, \vec{\partial} \ln |\vec{b}_{1t} - \vec{b}_t| \, \psi(b^-, \vec{b}_t)$$

at $y^+ \to \infty$, $F^{\mu\nu} \to 0$. $A_\mu \to 0$ up to gauge transformation

$$A_\mu = U \cancel{A_\mu}^{\,0} U^{-1} - \frac{i}{g_s} U \nabla_\mu U^{-1}$$

is a pure gauge. if $U = e^{-ig_s \Phi}$, then

$$A_\mu = \nabla_\mu \Phi$$

Therefore, we can write (2) as

$$(2) = \cdots \frac{ig_s}{2\pi} \int d\vec{b}_{1t} \, \cancel{\vec{\nabla}}_{1t} \Phi(\infty, \vec{b}_{1t}) \, \cancel{\vec{\nabla}}_t \ln|\vec{b}_{1t} - \vec{b}_t| \, \psi(b^-, \vec{b}_t) \cdots$$

$$= \cdots -\frac{ig_s}{2\pi} \int d\vec{b}_{1t} \, \Phi(\infty, \vec{b}_{1t}) \, \underbrace{\vec{\nabla}^2 \ln|\vec{b}_{1t} - \vec{b}_t|}_{= (2\pi)\, \delta(\vec{b}_{1t} - \vec{b}_t)} \, \psi(b^-, \vec{b}_t)$$

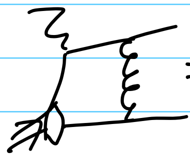
$$= \cdots -ig_s \Phi(\infty, \vec{b}_t) \psi(b^-, \vec{b}_t)$$

since $A_\mu = \nabla_\mu \Phi$

$$= \cdots +ig_s \int_{\vec{b}_t}^{\infty} \vec{A}_t(\infty, \vec{b}_{1t}) \cdot d\vec{b}_{1t} \quad \psi(b^-, \vec{b}_t)$$

$$= \cdots -ig_s \int_{\vec{b}_t}^{\infty} db^\mu_{1t} \, A_\mu(\infty, \vec{b}_{1t}) \, \psi(b^-, \vec{b}_t)$$

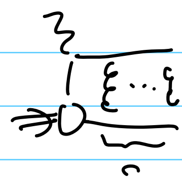
Put everything together, we find



$$= \cdots \left[-ig_s \int_{\vec{b}_t}^{\infty} db_{1t}^{\mu} A_\mu(\infty, \vec{b}_{1t}) + ig_s \int_{b^-}^{\infty} db_1^{-\mu} A_\mu(b_1^-, \vec{b}_t) \right] \psi(b^-, \vec{b}_t)|P\rangle$$

Belitsky, Ji. Yuan. hep-ph/0208038

$\Longrightarrow$

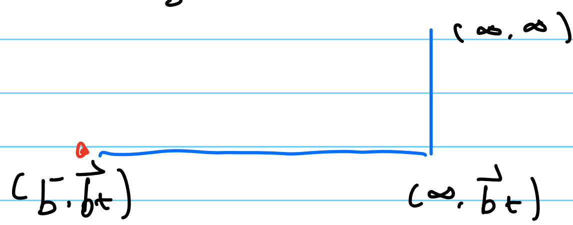
$\sum_n$ 

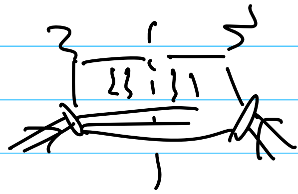
$$= \cdots W[\gamma] \, \psi(b^-, \vec{b}_t)|P\rangle$$

$$W[\gamma] = P\exp\left[-ig_s \int_{\vec{b}_t}^{\infty} db_{1t}^{\mu} A_\mu(\infty, \vec{b}_{1t})\right]$$

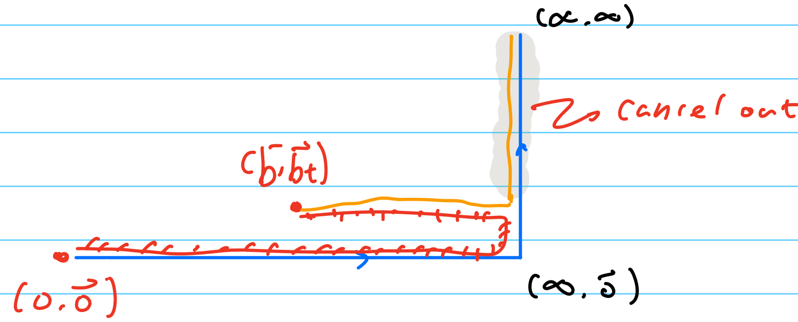
$$\times P\exp\left[ig_s \int_{b^-}^{\infty} db_1^{-\mu} A_\mu(b_1^-, \vec{b}_t)\right]$$

$=$




$\sim \langle p | \bar{\psi}(b^-, \vec{b}_t)\, W^\dagger[\gamma']\, W[\gamma]\, \psi(0) | P \rangle$

$W^\dagger[\gamma']\, W[\gamma] \Longrightarrow$



Staple-Shaped Wilson line in SIDIS:

the orientation is determined by "$-i0$" pole in the propagator, $\Longrightarrow$ "Final State Wilson line"

+ Staple-shaped wilson in SIDIS (out-going parton) its direction determined by the "$i0$"-pole. $W_{\sqsupset}$

+ $b_t \to 0$, reduce to collinear PDF wilson line with finite length.


+ Similar staple shaped wilson line exists in Drell-Yan but now it is the in-coming parton
  - $\Longrightarrow$ flipped "$i0$"-pole $\Leftarrow$ time-reversal.
  - $\Longrightarrow$ flipped orientation of the Wilson line $W_{\sqsubset}$
  - $\Longrightarrow$ different TMD PDFs in SIDIS & Drell-Yan
  - $\Longrightarrow$ Sivers effect: $f_{1T}^{\perp}{}_{\text{SIDIS}} = - f_{1T}^{\perp}{}_{\text{Drell-Yan}}$

Polarized TMDs

$$\Phi_{\alpha\beta}[p,k,S] = \int db\, e^{-ib\cdot k} \langle PS|\, \bar{\psi}_\beta(b^\mu) \cdots \psi_\alpha(0)\,|PS\rangle$$

p : proton momentum $\quad p^\mu = p^+ \frac{n^\mu}{2} + \frac{m^2}{2p^+}\frac{\bar{n}^\mu}{2}$.

S : Proton spin polarization vector

in the proton rest frame $\quad S^\mu = (0, \cos\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

e.g. $\quad |\uparrow\rangle \Rightarrow S^\mu = (0,0,0,1)$

$\quad |\downarrow\rangle \Rightarrow S^\mu = (0,0,0,-1)$

the spinor can be written by

e.g.

$$\psi_\alpha = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \chi_\alpha \\ \sqrt{p\cdot\bar{\sigma}}\ \chi_\alpha \end{pmatrix} \quad \text{with} \quad \chi_\alpha = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix}$$

In the proton boost frame

$$S^\mu = \frac{S_L}{m}\,\underbrace{\frac{p^+ n^\mu - p^- \bar{n}^\mu}{2}} + S_T^\mu$$

S_L : longitudinal spin

S_T : transverse spin

$$S\cdot S = -S_L^2 - S_T^2 = -1$$

ψ : Collinear field. "good component" *

for a given spinor $\xi(k)$ with $k \sim \frac{\bar{n}\cdot k}{2} n^\mu + O(\lambda)$

$\psi \equiv \frac{\not{n}\bar{\not{n}}}{4}\xi$ is the good component, $\bar{\psi} = \bar{\xi}\frac{\bar{\not{n}}\not{n}}{4}$, $\quad n = (1,0,0,1)$, $\bar{n} = (1,0,0,-1)$

easy to check that, $\not{n}\psi = 0$

$\Rightarrow$

$\bar{\psi}\not{n}\psi = 0$, $\bar{\psi}\gamma_\perp^\mu\psi = 0$. $\bar{\psi}\gamma^5\psi = 0$, $\bar{\psi}\psi = 0$

while $\bar{\psi}\bar{\not{n}}\psi$ ✓ $\quad \bar{\psi}\gamma^5\bar{\not{n}}\psi$ ✓. $\bar{\psi}\sigma_{\mu\nu}\bar{\not{n}}\psi = \bar{\psi}\gamma_t^i\bar{\not{n}}\psi$

one can decompose $\Phi_{\alpha\beta}$ in terms of the complete basis

$$\Gamma_i = 1,\ \gamma_5,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ \sigma^{\mu\nu} \text{ or } \sigma^{\mu\nu}\gamma_5$$

Note that $\quad \mathrm{Tr}[\Gamma_i \Gamma_j] = 4\delta_{ij}$

$$\Rightarrow \Phi_{\alpha\beta} = \frac{1}{4}\sum_{i=1} A_i(p,k,s)\,\Gamma_i \quad \text{with } \underbrace{A_i}_{\text{non-perturbative}} = \mathrm{Tr}[\Phi\Gamma_i]$$

Q: Which Γ_i is allowed at leading Power?

what is the general Form For $A_i(p,k,s)$?

good component $\rightarrow$ $\bar{\psi}\ldots\gamma^+\ldots\psi\ \ \gamma^-/2$

$$\Rightarrow \Gamma_i = \underline{\gamma^-} \equiv \gamma^\mu \hat{p}_\mu = \gamma^0 - \gamma^3,\ \ \underline{\gamma^-\gamma_5},\ \ \underline{\gamma_t^\mu \gamma^- \text{ or } \gamma_t^\mu\gamma^-\gamma_5}$$

$$\Phi[p,k,s] = A\gamma^- + B\gamma^-\gamma_5 + C_\mu \gamma_t^\mu \gamma^-\gamma_5 + iD_\mu \gamma_t^\mu\gamma^-$$

- at most 1 power of s^μ. $\quad u\bar{u} = \frac{\not{p}+M}{2}(\underset{\text{unpol.}}{1} + \underset{\text{p.l.}}{\gamma_5\not{s}})$ Proton is spin-1/2

- Hermiticity : $\Phi^\dagger[p,k,s] = \gamma^0\Phi[p,k,s]\gamma^0$

- Parity : $\Phi[p,k,s] = \gamma^0\,\Phi[\bar{p},\bar{k},-\bar{s}]\gamma_0 \qquad \bar{p} \equiv (p^0, -\vec{p})$

- Time-reversal : $\Phi^*(p,k,s) = (-i\gamma^5 C)\,\Phi(\bar{p},\bar{k},\bar{s})\,(-i\gamma^5 C)$

$C = i\gamma^2\gamma^0$

due to the Wilson line, time-reversal does not put constraint on the form of the coefficients

Proof of Hermiticity

$$\Phi^{\dagger}_{\alpha\beta} = \Phi^{*}_{\beta\alpha} = \int db\, e^{ib\cdot k}\langle PS|\bar{\psi}_\alpha(b)\cdots\psi_\beta(0)|PS\rangle^{*}$$

$$= \int db\, e^{ib\cdot k}\langle PS|\bar{\psi}_\alpha(0)\cdots\psi_\beta(-b)|PS\rangle^{*}$$ ← translational invariance

$$= \int db\, e^{-ib\cdot k}\langle PS|\bar{\psi}_\alpha(0)\cdots\psi_\beta(b)|PS\rangle^{*}$$ ← $b \to -b$

$$= \int db\, e^{-ib\cdot k}\langle PS|\left(\psi^{*}_\rho(0)\,\gamma^0_{\rho\alpha}\right)^{*}\cdots\psi^{*}_\beta(b)|PS\rangle$$ ← $\bar{\psi} = \psi^{\dagger}\gamma^0$

$$= \int db\, e^{-ib\cdot k}\langle PS|\psi_\rho(0)\,\gamma^0_{\rho\alpha}\cdots\psi^{*}_\beta(b)|PS\rangle$$ ← γ^0 real presentation

$$= \int db\, e^{-ib\cdot k}\langle PS|\psi^{*}_\beta(b)\cdots\psi_\rho(0)|PS\rangle\,\gamma^0_{\rho\alpha}$$ ← space-like $b^2 = -b_t^2$

$$= \int db\, e^{-ib\cdot k}\langle PS|\underbrace{\psi^{*}_\sigma(b)\,\gamma^0_{\sigma\tau}}_{\bar{\psi}}\,\gamma^0_{\tau\beta}\cdots\psi_\rho(0)|PS\rangle\,\gamma^0_{\rho\alpha}$$ ← $\gamma_0\gamma_0 = 1$

$$= \gamma^0_{\beta\tau}\int db\, e^{-ib\cdot k}\langle PS|\bar{\psi}_\tau(b)\cdots\psi_\rho(0)|PS\rangle\,\gamma^0_{\rho\alpha}$$ ← symmetric γ^0

$$= \gamma^0_{\beta\tau}\,\Phi_{\tau\rho}(P,k,S)\,\gamma^0_{\rho\alpha}$$

- Proof of Parity

note that under parity $p^\mu \to \bar{p}^\mu$. $S^\mu \to -\bar{S}^\mu$
$b^\mu \to \bar{b}^\mu$, $P\psi(b)P^{-1} = \gamma_0 \psi(\bar{b})$

$$\Phi_{\alpha\beta}(p,k,S) = \int db\, e^{-ib\cdot k} \langle PS| \bar{\psi}_\beta(b) \cdots \psi_\alpha(0) |PS\rangle$$

$$= \int db\, e^{-ib\cdot k} \langle PS| P^{-1} P\, \bar{\psi}_\beta(b) P^{-1} \cdots P\, \psi_\alpha(0) P^{-1} P |PS\rangle$$

$$= \int db\, e^{-ib\cdot k} \langle \bar{P}, -\bar{S}| \bar{\psi}_\sigma(\bar{b})\gamma^0_{\sigma\beta} \cdots \gamma^0_{\alpha\rho} \psi_\rho(0) |\bar{P}, -\bar{S}\rangle \leftarrow \begin{aligned} &P\psi^\dagger\gamma^0 P^{-1} \\ &= \psi^\dagger\gamma^0\gamma^0 \\ &= \bar{\psi}\gamma^0 \end{aligned}$$

$$= \gamma^0_{\alpha\rho} \int db\, e^{-i\bar{b}\cdot\bar{k}} \langle \bar{P}, -\bar{S}| \bar{\psi}_\sigma(\bar{b}) \cdots \psi_\rho(0) |\bar{P}, -\bar{S}\rangle \gamma^0_{\sigma\beta} \quad \leftarrow b\cdot k = \bar{b}\cdot\bar{k}$$

$$= \gamma^0_{\alpha\rho}\, \Phi_{\rho\sigma}(\bar{p}, \bar{k}, -\bar{S})\, \gamma^0_{\sigma\beta} \quad \leftarrow b \to \bar{b}$$

- Constraint from Hermiticity $\quad \Phi^{\dagger} = \gamma^{0} \Phi \gamma^{0}$

$$\Phi[p.k,s] = A\gamma^{-} + B\gamma^{-}\gamma_5 + C_\mu \gamma_t^{\mu} \gamma^{-}\gamma_5 + iD_\mu \gamma_t^{\mu} \gamma^{-}$$

+ $A\gamma^{-}$ term

$$LHS: (A\gamma^{-})^{\dagger} = A^{*}(\gamma^{0}-\gamma^{3})^{\dagger} = A^{*}(\gamma^{0}+\gamma^{3}) = A^{*}\gamma^{+}$$

$$\vec{\gamma}^{\dagger} = (-\vec{\tau}\,\vec{\sigma})^{\dagger} = (+\vec{\tau}\,{}^{-}\vec{\sigma}) = -\vec{\gamma}$$

$$RHS: \gamma^{0}(A\gamma^{-})\gamma^{0} = A\gamma^{0}(\gamma^{0}-\gamma^{3})\gamma^{0} = A(\gamma^{0}+\gamma^{3}) = A\gamma^{+}$$

$$LHS = RHS \Longrightarrow A = A^{*} \Longrightarrow \boxed{A \text{ is real}}$$

+ $B\gamma^{-}\gamma_5$ term

$$LHS: (B\gamma^{-}\gamma_5)^{\dagger} = B^{*}\gamma_5\gamma^{+} = -B^{*}\gamma^{+}\gamma_5$$

$$RHS: \gamma^{0}B\gamma^{-}\gamma^{5}\gamma^{0} = -B\,\gamma^{0}\gamma^{-}\gamma^{0}\gamma^{5} = -B\gamma^{+}\gamma^{5}$$

$$LHS = RHS \Longrightarrow \boxed{B \text{ is real}}$$

+ $C_\mu \gamma_t^{\mu}\gamma^{-}\gamma_5$ term

$$LHS: (C_\mu \gamma_t^{\mu}\gamma^{-}\gamma_5)^{\dagger} = C_\mu^{*}\gamma_5\gamma^{+}(-\gamma_t^{\mu}) = +C_\mu^{*}\gamma_t^{\mu}\gamma^{+}\gamma_5$$

$$RHS: \gamma_0 C_\mu \gamma_t^{\mu}\gamma^{-}\gamma_5\gamma_0 = C_\mu \gamma_t^{\mu}\gamma_0\gamma^{-}\gamma_0\gamma_5 = C_\mu \gamma_t^{\mu}\gamma^{+}\gamma_5$$

$$LHS = RHS \Longrightarrow \boxed{C_\mu \text{ is real}}$$

+ $\boxed{D_\mu \text{ is real.}}$

- Constraint from Parity $\quad \Phi(p,k,S) = \gamma^0\, \Phi(\bar{p},\bar{k},-\bar{S})\,\gamma^0$

+ $A\gamma^-$ term

LHS : $\quad A(p,k,S)\,\gamma^-$

RHS : $\quad A(\bar{P},\bar{k},-\bar{S})\,\gamma_0\gamma^+\gamma_0 \;=\; A(\bar{P},\bar{k},-\bar{S})\,\gamma^-$

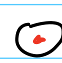
since $\gamma^- = \gamma_\mu \hat{p}^\mu \xrightarrow{\;\mathcal{P}\;} \gamma_\mu \hat{\bar{p}}^\mu = \gamma^+$

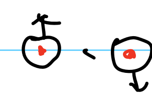
LHS = RHS $\Longrightarrow$ $A(p,k,S) = A(\bar{p},\bar{k},-\bar{S})$ is parity even

$\Longrightarrow$ need even # of momentum $p,\ \bar{p},\ \vec{k}_T$
+ at most 1 power of S
+ scalar + $p^2 = \bar{p}^2 = 0$

$\vec{p}\cdot\dfrac{\vec{k}_t\times\vec{S}_t}{m}$

$$\Longrightarrow A(p,k,S) = \frac{1}{2}\Big\{\underbrace{f_1(k\cdot\bar{p},\,k_T^2)}_{\text{unpol quark, unpol proton}} - \underbrace{\epsilon_T^{\rho\sigma}}_{\epsilon^{\rho\sigma\mu\nu}\hat{p}_\mu\hat{\bar{p}}_\nu}\frac{k_{T\rho}S_{T\sigma}}{m}\underbrace{f_{1T}^{\perp}(k\cdot\bar{p},\,k_T^2)}_{\text{unpol. quark, T-Pol. proton, Sivers}}\Big\}$$





+ $B\gamma_5\gamma^-$ term

LHS : $\quad B(p,k,S)\,\gamma_5\gamma^-$

RHS : $\quad B(\bar{p},\bar{k},-\bar{S})\,\gamma_0\gamma_5\gamma^+\gamma_0 \;=\; -B(\bar{p},\bar{k},-\bar{S})\,\gamma_5\gamma^-$

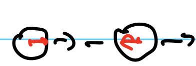
LHS = RHS $\Longrightarrow$ $B(p,k,S)$ is Parity odd

$\Longrightarrow$ odd # of momentum + 1 S + scalar

$$\Longrightarrow B(p,k,S) = \frac{1}{2}\Big\{\underbrace{S_L}_{\sim p\cdot S}\, g_1 - \frac{k_T\cdot S_T}{m}\, g_{1T}^{\perp}\Big\}$$

g_1: L-Pol. proton, L-Pol. quark

$g_{1T}^{\perp}$: T. Pol. Proton, L. Pol. quark



+ $C_\mu \gamma_t^\mu \gamma^- \gamma^5$ term

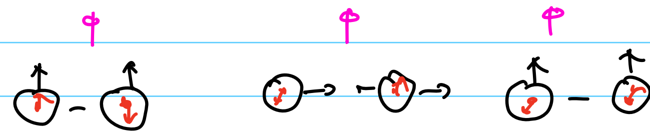
LHS : $C_\mu(p.k.S)\, \gamma_t^\mu \gamma^- \gamma^5$

RHS : $C_\mu(\bar{p}.\bar{k}.-\bar{S})\, \gamma_0 \gamma_t^\mu \gamma^+ \gamma^5 \gamma_0 \;=\; C_\mu(\bar{p}.\bar{k}.-\bar{S})\, \gamma_t^\mu \gamma^- \gamma^5$

$\Rightarrow$ C_μ is parity even

$\Rightarrow$ even # of momentum + 1 S + Vector coupled to γ_t^μ

$\Rightarrow$ $C_\mu = \frac{1}{2}\left\{ h_1 S_{T\mu} + \left(S_L h_{1L}^\perp - \frac{k_T \cdot S_T}{M} h_{1T}^\perp\right) k_{T\mu} \right\}$



+ $i D_\mu \gamma_t^\mu \gamma^-$ term

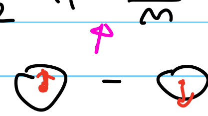
LHS : $i D_\mu(p.k.S)\, \gamma_t^\mu \gamma^-$

RHS : $i D_\mu(\bar{p}.\bar{k}.-\bar{S})\, \gamma_0 \gamma_t^\mu \gamma^+ \gamma_0 = -i D_\mu(\bar{p}.\bar{k}.-\bar{S})\, \gamma_t^\mu \gamma^-$

$\Rightarrow$ D_μ is parity odd

$\Rightarrow$ odd # of momentum + 1 S + Vector coupled to γ_t^μ

$\Rightarrow$ $D_\mu = \frac{1}{2} h_1^\perp \frac{k_{T,\mu}}{M}$



Hence

$$\Phi = \frac{1}{2}\Big\{ f_1 \gamma^- - f_{1T}^{\perp} \epsilon_T^{\rho\sigma} \frac{k_{T\rho}}{M} S_{T\sigma} \gamma^-$$
$$+ \left(S_L g_1 - \frac{k_T\cdot S_T}{M} g_{1T}^{\perp}\right)\gamma_5\gamma^-$$
$$+ h_1 \not{S}_T \gamma^- \gamma_5 + \left(S_L h_{1L}^{\perp} - \frac{k_T\cdot S_T}{M} h_{1T}^{\perp}\right)\frac{\not{k}_T \gamma^- \gamma_5}{M}$$
$$+ i h_1^{\perp} \frac{\not{k}_T \gamma^-}{M} \Big\}$$

8 TMDs.

+ Some of them are naively T-odd: $f_{1T}^{\perp}$, $h_1^{\perp}$

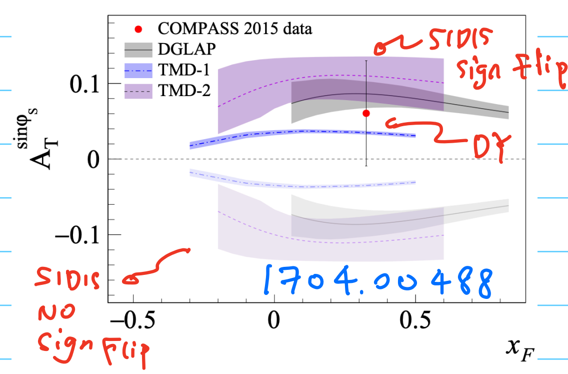
note under time-reversal $p \to \bar{p}$. $k_t \to \bar{k}_t$, $S \to \bar{S}$

vanishes under naive factorization. $f_{1T}^{\perp} = - f_{1T}^{\perp} \Rightarrow f_{1T}^{\perp} = 0$

+ but also switches Wilson line $\underset{\text{SIDIS}}{W_J} \leftrightarrow \underset{\text{Drell-Yan}}{W_C}$

$\Rightarrow$ $(f_{1T}^{\perp})_{\text{SIDIS}} = -(f_{1T}^{\perp})_{\text{Drell-Yan}}$ → observable effects

$(h_1^{\perp})_{\text{SIDIS}} = -(h_1^{\perp})_{\text{Drell-Yan}}$



+ Similar decomposition for Quark TMD FF & gluon TMDs.

Proton Polarization	Parton Polarization: unpol.	longitudinal	transverse
U	$f_1 =$		$h_1^{\perp} = \; - \;$ Boer-Mulders
L		$g_1 = \; - \;$ helicity	$h_{1L}^{\perp} = \; - \;$ worm-gear
T	$f_{1T}^{\perp} = \; - \;$ Sivers	$g_{1T}^{\perp} = \; - \;$ Worm-gear	$h_1 = \; - \;$ transversity $h_{1T}^{\perp} = \; - \;$ Pretzelosity