Introduction to perturbative QCD

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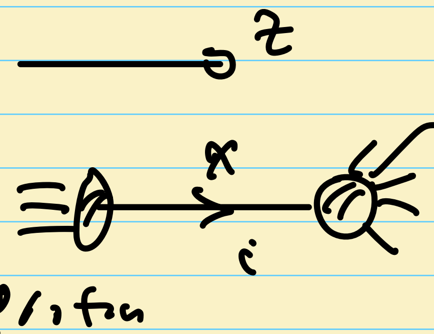
Part 4. Collinear Factorization
& parton distribution function

$\Longrightarrow$ Initial state singularity in ep & pp collisions

$\Longrightarrow$ pdf & DGLAP Evolution

— initial state singularity.

In the introduction part, we have mentioned that for proton initiating processes



$$= f_{i/p}(x) \cdot \hat{\sigma}_i$$

where $f_{i/p}$ is the parton distribution function, i.e. the prob. to select a parton i out of a proton. Here

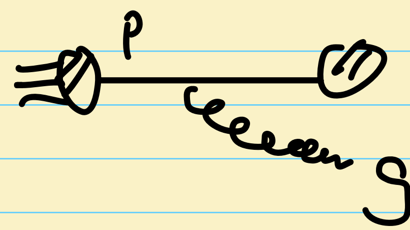
$$x \equiv \frac{\bar{n}\cdot p}{\bar{n}\cdot P_{\text{proton}}} \simeq \frac{E_p}{E_{\text{proton}}}$$

is the momentum fraction, $\bar{n}^\mu = (1,0,0,-1)$

and $p = \frac{\bar{n}\cdot p}{2}(1,0,0,1) \equiv \frac{\bar{n}\cdot p}{2} n^\mu$

This is called naive parton model. it will recieve higher order corrections.

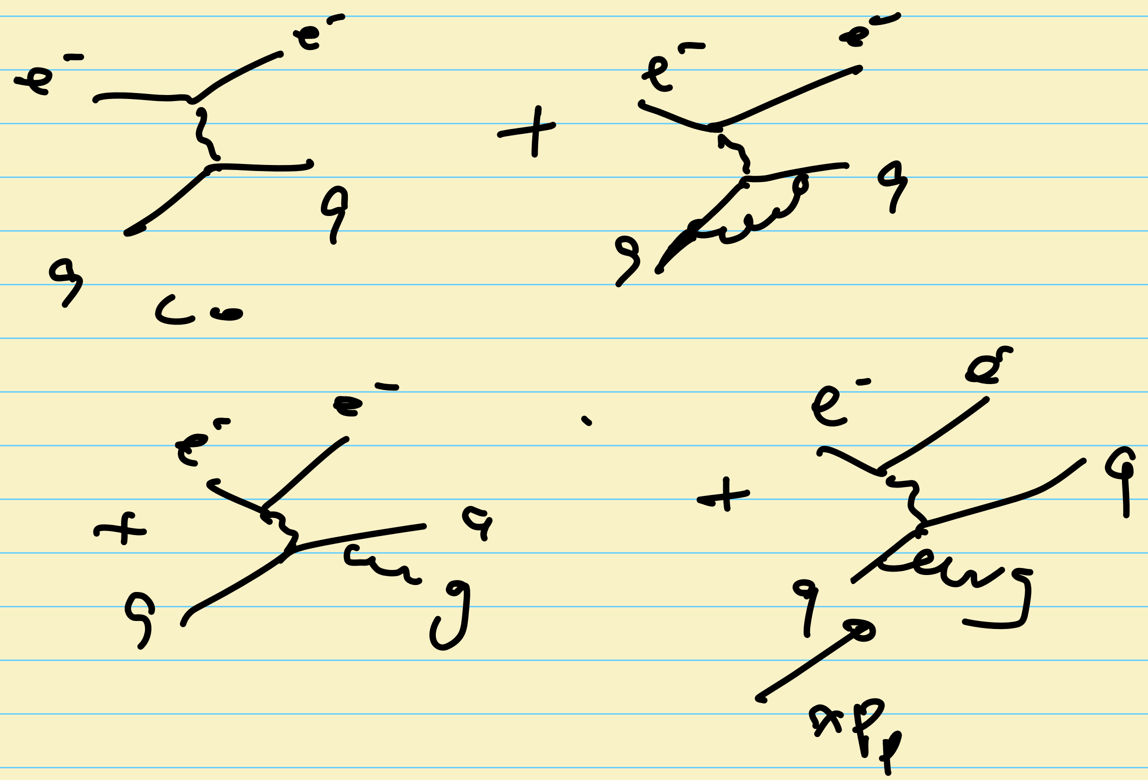
for instance



we know from previous Lectures that when $g \to 0$. or/and $g \parallel p$. it will generate poles. when the partons are in the final state. we have seen that these poles will cancel against the Virtual corrections to give finite result for IR safe quantities.

and we will see the issue with the cancellation when we have an incoming parton in the initial state

To highlight, we consider the partonic process $e^- q \to e^- X$, which is nothing but replacing the proton with a quark in DIS



$+ \cdots$

This is can be obtained by $e^+ e^- \to X$ through crossing $(e^+ \to e^-, \bar{q} \to q)$

explicit calculation finds

$$G^{(1)}_{real} = \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} (1-\epsilon) \left(\frac{\mu^2}{Q^2}\right)^{\epsilon}$$

$$\times \left\{ \frac{2}{\epsilon^2} \delta(1-z) - \frac{1}{\epsilon} \frac{1+z^2}{(1-z)_+} + \frac{3}{2\epsilon} \delta(1-z) \right.$$

$$\left. + \text{ finite terms} \right\}$$

$$G^{(1)}_{virt.} = \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8\right) \delta(1-z)$$

where

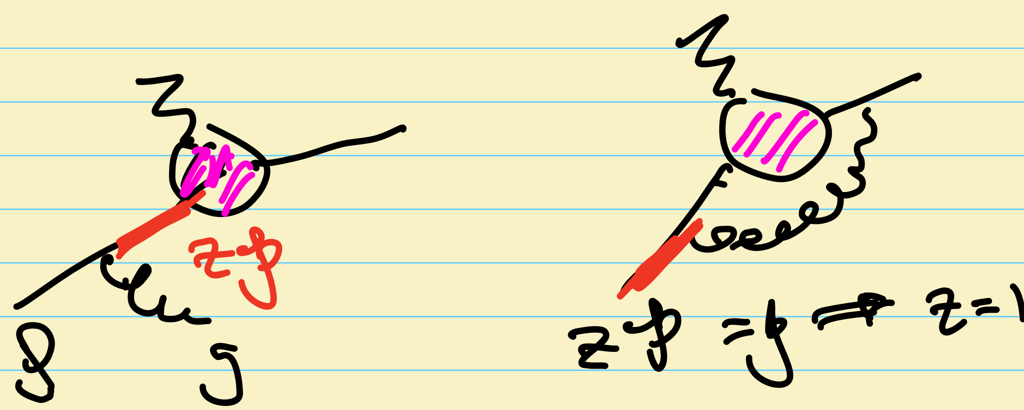
$$z = \frac{x_B}{x} = -\frac{q^2}{x\, 2p_p \cdot q} = -\frac{q^2}{2p \cdot q}$$

and x_B is the Bjorken-x

$$x_B = -\frac{q^2}{2p_p \cdot q}$$

where q is momentum of the virtual photon. $p = x p_p$ is the initial quark momentum

Therefor, z is the partonic Bjorken-x variable, i.e. the momentum fraction of the quark that participates "directly" the hard interaction:



$\Rightarrow$ in tree, virtual & final state radiation

$$z=1$$

can NOT cancel each other!!

in initial state radiation

$$z \leq 1$$

only soft radiation gives $z=1$

add up real & virtual. we find

$$\hat{G}^{(1)} = \frac{\alpha_s}{2\pi} C_F \times \left\{ \underline{-\frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+} - \frac{3}{2\epsilon}\delta(1-z)} \right.$$

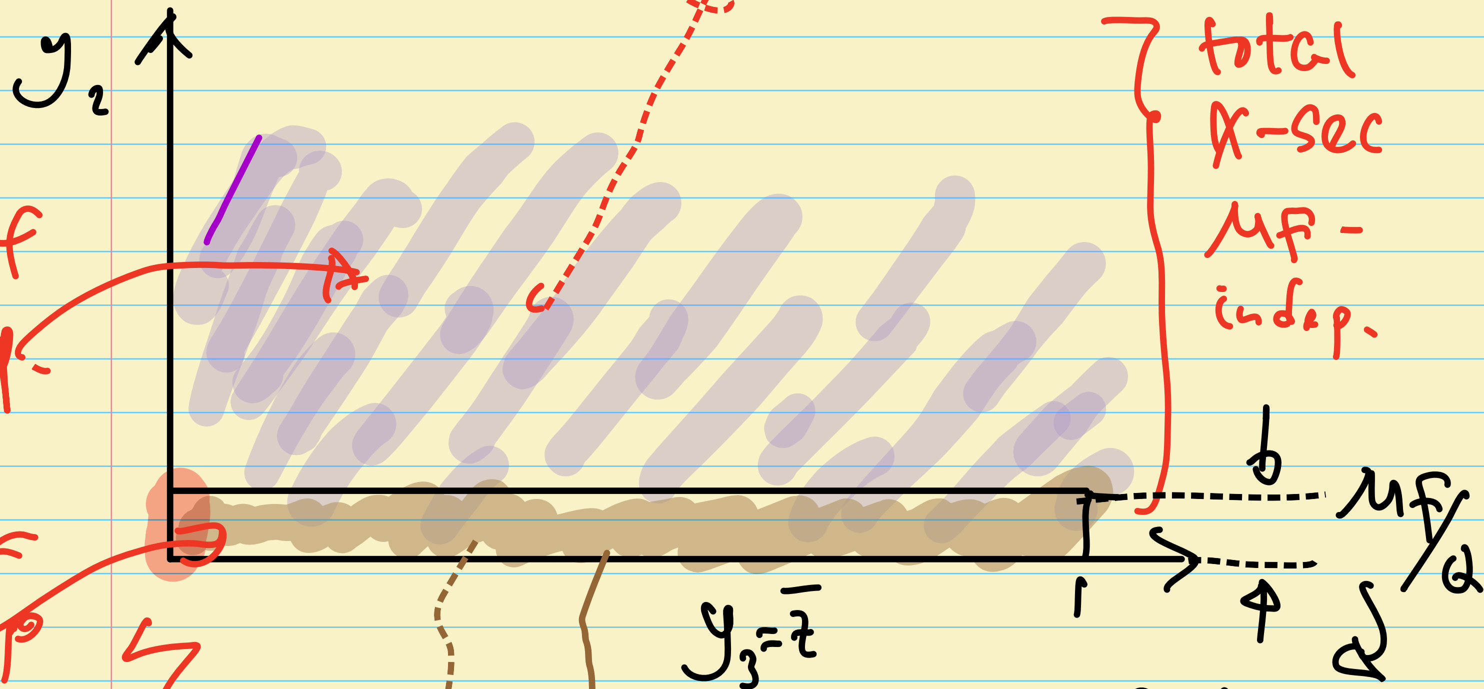
$$\left. + \text{finite terms} \right\}$$

$$= \frac{\alpha_s}{2\pi} C_F \times \left\{ \underline{\frac{-1}{\epsilon} P_{qq}(z)} + \dots \right\}$$

pole does not cancel !?!
It tells us that we are probing
an exclusive N.P. initial state
arises due to removing the
λ_{QCD} cutoff in the infrared.

Similar to Jets discussed before!!

H "hard part ("3-jet")



total X-sec μ_f-indep.

μ_f dep.

μ_f dep.

μ_f indep.

Subtract soft

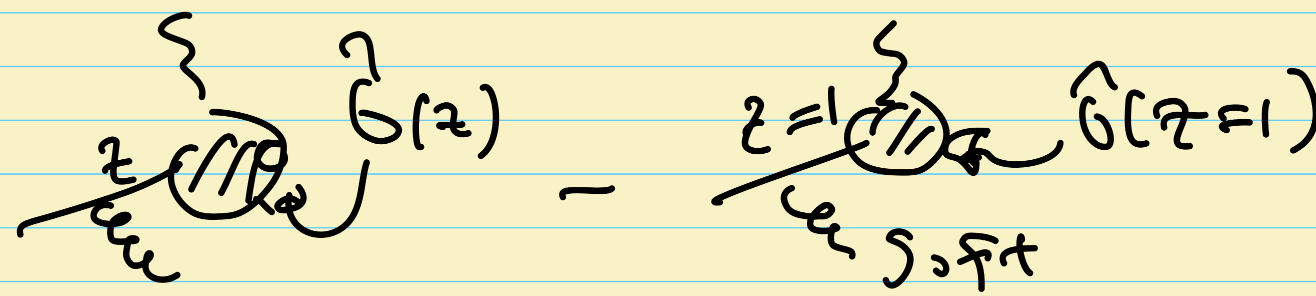
Cutoff to define "collinear"

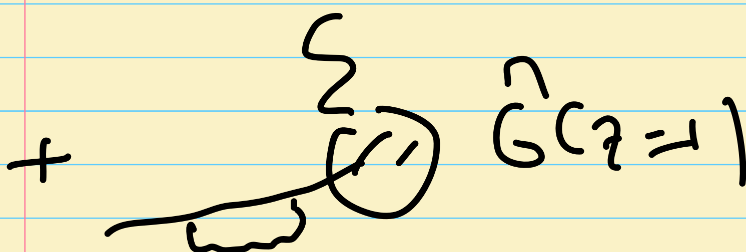
Collinear radiation that should be regarded as part of the initial proton wave function & need to be removed from the hard interaction

Collinear part ("2-jet")

the poles originate from the miss match between initial collinear radiation and the collinear virtual correction, which should be regarded as part of the proton wave function, and should be removed from the hard process.

like before the collinear part can be obtained by the splitting function (now it is the space-like splitting function. at $\mathcal{O}(\alpha_s)$ it is identical to the splitting function we found previously)





virtual pole $\sim \frac{3}{2\epsilon}\,\delta(1-z)$

Collinear

$$\sim \frac{\alpha_s}{2\pi} C_F \int_0^{\mu_F^2/\hat{Q}^2} dy_2\, y_2^{-1-\epsilon} \int_0^1 \frac{dz}{z} \frac{1+z^2}{(1-z)^{1+\epsilon}} \left[\hat{\sigma}(z) - \hat{\sigma}(1)\right]$$

+ Virtual initial collinear pole

$$= \frac{\alpha_s}{2\pi} C_F \frac{-1}{\epsilon} \left(\frac{\mu_F^2}{Q^2}\right)^{-\epsilon} \int_0^1 \frac{dz}{z}$$

$$\times \left\{ \frac{1+z^2}{(1-z)_+} + \mathcal{O}(\epsilon) \right\} \hat{\sigma}(z)$$

+ Virtual initial collinear pole

$$= \frac{\alpha_s}{2\pi} C_F \int \frac{dz}{z} \left\{ -\frac{1}{\epsilon} P_{qq}(z) + \ln\frac{\mu_F^2}{Q^2} P_{qq}(z) + \dots \right\}$$

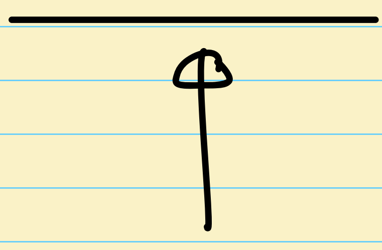
therefore when remove the collinear from the hard, we find the remaining is finite but depends on the artificial cuttoff μ_F through $\ln \frac{\mu_F}{Q} P_{qq}$

Since the total x-sec is μ_F independent, therefore, we conclude the the pdf will depends on μ_F (how many radiation is categorized into the proton & how many into the hard)

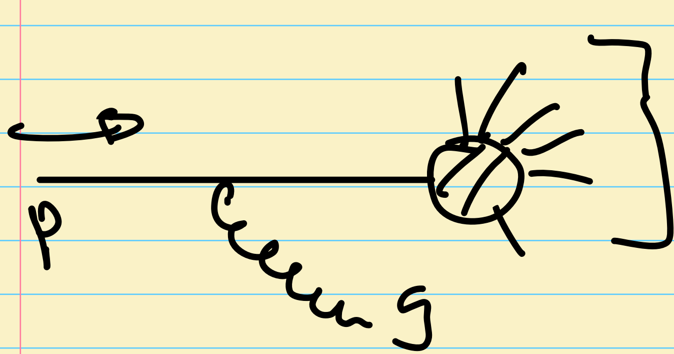
$\frac{df_i}{d\ln\mu_F} \sim \frac{\alpha_s}{2\pi} P_{ij} \otimes f_j$

that sums up radiations (l.gr) from Λ_{QCD} to the factorization scale μ_F

$\mu_F \sim Q$



Λ_{QCD} proton

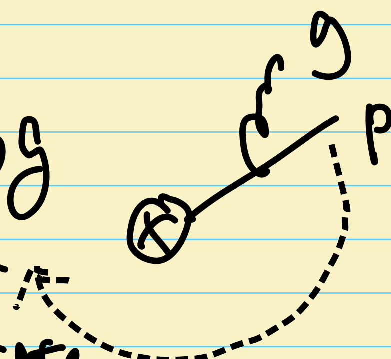


Inclusive enough

Focus on the collinear singularity and therefore $g \parallel p$

$$z = \frac{\bar{n}\cdot g}{\bar{n}\cdot p}$$

= crossing p to the initial state



$$x = \frac{\bar{n}\cdot p}{\bar{n}\cdot p + \bar{n}\cdot g} \to \frac{-\bar{n}\cdot p}{-\bar{n}\cdot p + \bar{n}\cdot g} = \frac{1}{1-z} = \frac{1}{z}$$

$$P(x) \to -P\left(\frac{1}{z}\right) = -\frac{1+1/z^2}{1-1/z} = \frac{1}{z} P(z)$$

$$(p-g)^2 = -2p\cdot g = -2\frac{\bar{n}\cdot p}{2} n g = -\bar{n}\cdot p \frac{\vec{g}^2}{\bar{n}\cdot g} = -\frac{\vec{g}^2}{\bar{z}}$$

$$\int \frac{d^d g}{(2\pi)^{d-1}} \delta(g^2)$$

$$= \frac{1}{2}\int \frac{d\bar{n}\cdot g \; dn\cdot g \quad d^{d-2}g_\perp}{(2\pi)^{d-1}} \delta(\bar{n}\cdot g\, n\cdot g - \vec{g}_\perp^2)$$

$$= \frac{1}{2}\frac{1}{(2\pi)^{d-1}}\int \frac{d\bar{n}\cdot g}{\bar{n}\cdot g} \; g^{d-3} dg \; \Omega_{d-2}$$

$$= \frac{1}{4}\frac{1}{(2\pi)^{d-1}} \int \frac{d\bar{n}\cdot g}{\bar{n}\cdot g} \; (g_\perp^2)^{\frac{d-4}{2}} dg_\perp^2 \; \Omega_{d-2}$$

$$= \frac{1}{4}\frac{\Omega_{d-2}}{(2\pi)^{d-1}} \int \frac{dz}{1-z} \int^{\mu_F} dg_\perp^2 \, (g_\perp^2)^{-\epsilon}$$

$$\times \quad -\frac{\bar{z}}{g_\perp^2} \cdot \frac{1}{z}\left(\frac{1+z^2}{1-z} \quad \cdots \right) g_s^2 C_F$$

$$= g_s^2 C_F \frac{1}{4} \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \int_0^{\mu_F} dg_\perp^2 \, (g_\perp^2)^{-1-\epsilon} \int \frac{dz}{z}\left(\frac{1+z^2}{1-z} + \cdots\right)$$