Introduction to perturbative QCD

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1.4.2025 @ Sun YAT-SEN Univ.

Part 3. Higher Order Corrections, Infra-red behaviors & Jets

$\Rightarrow$ NLO Correction to $e^+e^- \to X$

$\Rightarrow$ Soft & Collinear limit.

general features

$\Rightarrow$ Jets & IR safety

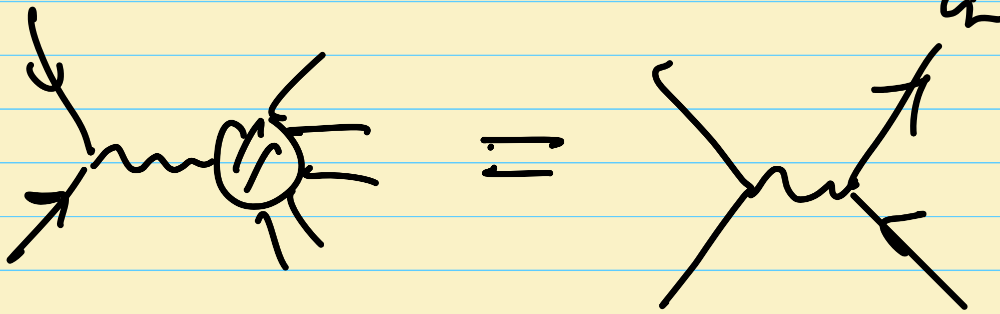
& large logarithms

High-order corrections, Infra-red behavior & Jets.

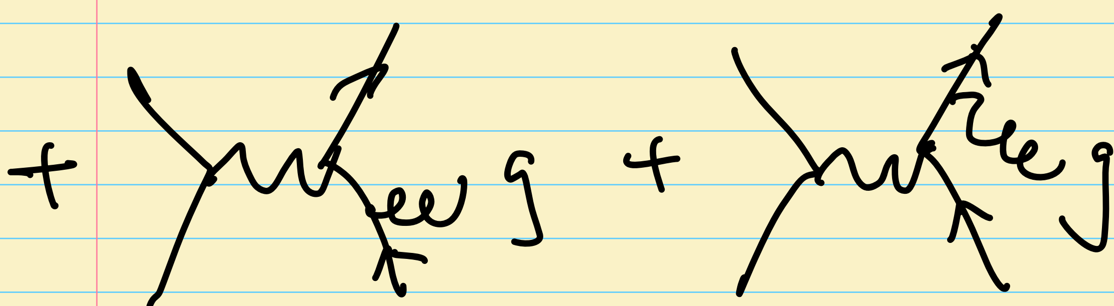
- NLO correction to $e^+e^- \to X$

Now we know that for large q^2. the hadron X-sec can be well approximated by pQCD calculation with partons

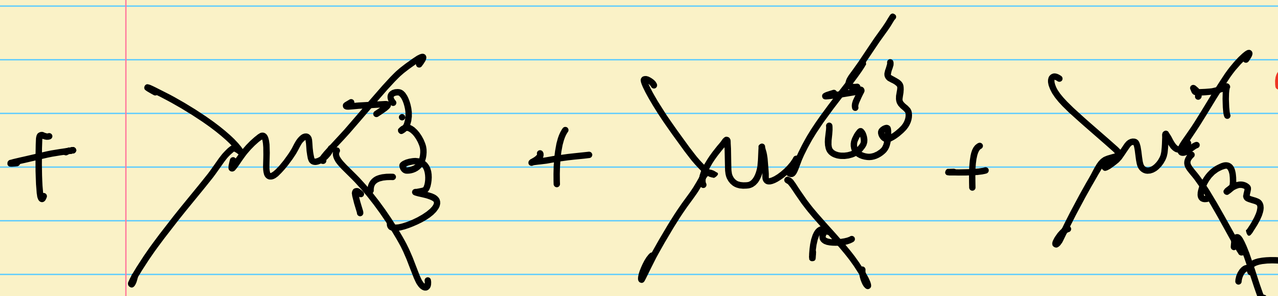
$$\sigma = \sigma^{(0)} + \frac{\alpha_s}{2\pi}\sigma^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2\sigma^{(2)} + \cdots$$



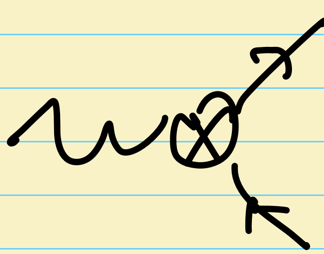
→ LO



→ NLO real



→ NLO virtual

$+$  $\left(Z_\Gamma Z_\psi - 1\right)$ → counter term

$+ \mathcal{O}(\alpha_s^2)$

$\Rightarrow$ Leading order

$$\underbrace{\sigma^{(0)}}_{} = \sum_q e_q^2 N_c \frac{4\pi\alpha^2}{3s}$$

obtained using optical theorem before

can also be calculated by phase-space integration

recall that

$$\sigma = \frac{4\pi\alpha}{q^2}\left(-\frac{1}{2}\right) H(q^2)$$

&

$$H(q^2) \equiv \frac{e_q^2}{3q^2} \sum_X |\langle 0|J^\mu|X\rangle|^2 (2\pi)^4 \delta^{(4)}(q-p_X)$$

at LO, $X = q, \bar{q}$

$$\underbrace{\sum_X (2\pi)^4 \delta(\cdots)}_{[dPS]_2} = \int \underbrace{\frac{d^4p}{(2\pi)^3}\delta(p^2)}_{\text{1-particle phase space}} \frac{d^4\bar{p}}{(2\pi)^3}\delta(\bar{p}^2)(2\pi)^4\delta^{(4)}(q-p-\bar{p})$$

$$[dPS]_2 = \frac{1}{4\pi^2}\int d^4p\, \delta(p^2)\, \delta(\underbrace{q^2 - 2p\cdot q}_{s})$$

$q^2 = s,\ 2p\cdot q = 2p\cdot(l+\bar{l})$

$= -u-t$

$$= \frac{1}{4\pi^2}\int d^4p\, \delta(p^2)\, \delta(s+t+u)$$

light-cone decomposition

$$l = \sqrt{s}/_2(1,0,0,1)\ ,\ \bar{l} = \sqrt{s}/_2(1,0,0,-1)$$

$$\Rightarrow\ p^\mu = \frac{p\cdot l}{l\cdot\bar{l}}\bar{l}^\mu + \frac{p\cdot\bar{l}}{l\cdot\bar{l}}\cdot l^\mu + p_\perp^\mu$$

$$= \frac{-t}{s}\bar{l}^\mu + \frac{-u}{s}l^\mu + p_\perp^\mu$$

$$\Rightarrow\ p^2 = \frac{-t}{s}\frac{-u}{s}s - \vec{p}_\perp^{\,2} = \frac{tu}{s} - \vec{p}_\perp^{\,2}$$

$$\frac{p\cdot\bar{l}}{l\cdot\bar{l}} = \frac{1}{s/_2}\frac{\sqrt{s}}{2}(p_0+p_3) = \frac{1}{\sqrt{s}}(p_0+p_3)$$

$$\frac{p\cdot l}{l\cdot\bar{l}} = \frac{1}{\sqrt{s}}(p_0-p_3)$$

$$\begin{vmatrix} \frac{1}{\sqrt{s}} & \frac{1}{\sqrt{s}} \\ \frac{1}{\sqrt{s}} & -\frac{1}{\sqrt{s}} \end{vmatrix} = \frac{2}{s}$$

$$\Rightarrow\ dp_0\, dp_3 = \frac{s}{2}\, d\left(\frac{-t}{s}\right) d\left(\frac{-u}{s}\right)$$

Therefore, we have

$$[dPS]_2 = \frac{1}{4\pi^2}\frac{s}{2}\int d\left(\frac{-t}{s}\right) d\left[\frac{-u}{s}\right) d^2P_t$$

$$\times\, \delta\left(\frac{tu}{s}-\vec{p}_\perp^{\,2}\right)\delta(s+t+u)$$

$\to \frac{1}{2}dp_\perp^2$

$$= \frac{1}{4\pi^2}\frac{s}{2}\int_0 d\left(\frac{-t}{s}\right) d\left[\frac{-u}{s}\right)\, P_t \underline{dP_t}\, d\varphi \quad \to 2\pi$$

$$\delta\left(\frac{tu}{s}-P_\perp^2\right)\times\frac{1}{s}\times\delta\left(1-\frac{-t}{s}-\frac{-u}{s}\right)$$

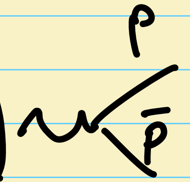
let $\left(\frac{-t}{s}\right) = x$ then $-u/s = 1-x$

$$\Rightarrow \boxed{[dPS]_2 = \frac{1}{8\pi}\int_0^1 dx}$$

In $d = 4-2\epsilon$ dim. the phase-space is

$$\frac{1}{4}\frac{\Omega_{d-2}}{(2\pi)^{d-2}}(q^2)^{-\epsilon}\int_0^1 dx\, x^{-\epsilon}(1-x)^{-\epsilon}$$

Where $\Omega_{d-2} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$ is the solid angle

The $|\langle 0|J^\mu|q\bar{q}\rangle|^2 = \Big|$ 

$\Big|^2$

$$= \sum_{\text{spin}} \sum_{\text{color}} ie\, \bar{u}\gamma^\mu u \quad (-ie)\, u \gamma_\mu \bar{u}$$

$$= e^2 \, \underset{\text{spin}}{\mathrm{Tr}} [\gamma^\mu \bar{\not{p}} \gamma_\mu \not{p}] \, \underset{\text{color}}{\mathrm{Tr}} (1)$$

$$= e^2 \, 4(\bar{p}^\mu p_\mu + p^\mu \bar{p}_\mu - g^\mu_\mu \, p\cdot\bar{p}) N_c$$

$$= e^2 4(q^2 - 2q^2) N_c = 4\pi\alpha \, (-4) q^2 N_c$$

for the NLO correction, we may also need $d = 4-2\epsilon$ dimension result, which is

$$e^2 4(q^2/2 + q^2/2 - d\, q^2/2) N_c$$

$$= 4\pi\alpha \, (-2)(d-2) q^2 N_c$$

$$\Rightarrow H^{(1)}(q^2) = \frac{e_q^2}{3q^2}\frac{1}{8\pi}\int_0^1 dx\ 4\pi\alpha(-4)\, q^2 N_c$$

$$= -2 e_q^2 N_c \frac{\alpha}{3}$$

$$\Rightarrow \sigma^{(0)} = \sum_{e_q} \frac{4\pi\alpha}{q^2}\left(-\frac{1}{2}\right)(-2)\, e_q^2 N_c \alpha/3$$

$$= \sum_{e_q} N_c e_q^2 \frac{4\pi\alpha^2}{3q^2}$$

reproduce
optical theorem
result

$\Rightarrow$ NLO correction

$$\frac{\alpha_s}{2\pi}\sigma^{(1)} = \frac{\alpha_s(\mu)}{2\pi} C_F \frac{3}{2}\sigma^{(0)}$$

and

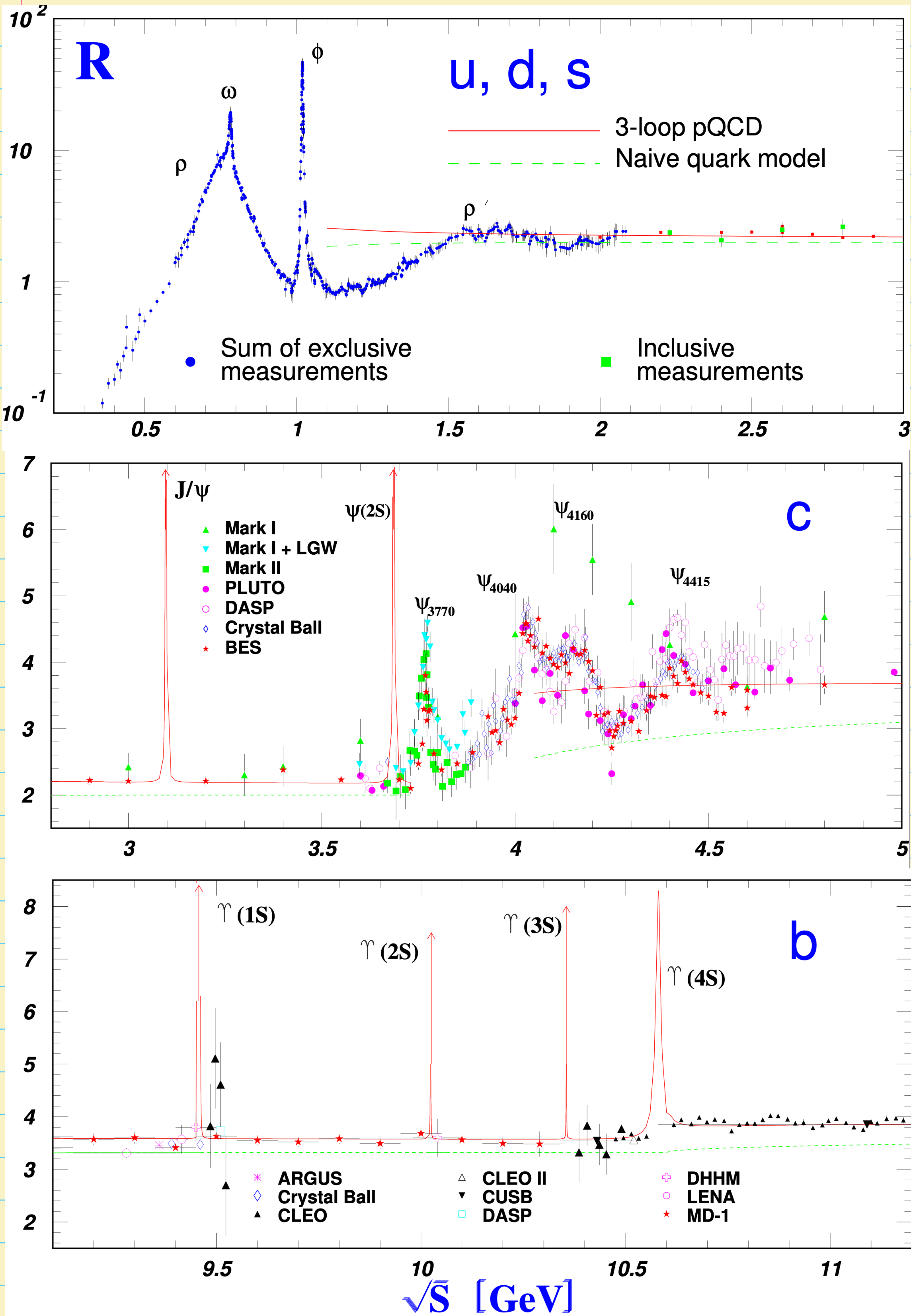
$$\frac{\alpha_s}{2\pi}\sigma^{(1)}_{virt,} = \sigma^{(0)}\frac{\alpha_s}{2\pi}C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\times\left\{-\frac{2}{\epsilon_{IR}^2}-\frac{3}{\epsilon_{IR}}-8+\frac{7}{6}\pi^2\right\}$$

$$\frac{\alpha_s}{2\pi}\sigma^{(1)}_{real} = \sigma^{(0)}\frac{\alpha_s}{2\pi}C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\times\left\{\frac{2}{\epsilon_{IR}^2}+\frac{3}{\epsilon_{IR}}+\frac{19}{2}-\frac{7}{6}\pi^2\right\}$$

All UV-poles have been cancelled here $1/\epsilon^i$ are IR-poles they cancel between virtual and real,

KLN Theorem

depends on the scale choice. $\boxed{\ln\frac{\mu}{Q}}$



hep-ph/0312114

sizable high-order corrections

* Sketch of the Calculation

- Virtual: *

1) dim-reg. $d = 4 - 2\epsilon$ to regulate both UV & IR divergence

feynman. Param., loop-integration

2) Vertex normalization

$$\bar{q}_0 \Gamma_0^\mu q_0 = Z_\Gamma Z_q \bar{q} \Gamma^\mu q$$

to remove UV divergence ($\overline{MS}$) and left with IR poles & finite terms

- real : **
  1) dim-reg. $d = 4 - 2\epsilon$ to regulate IR divergence.
  2) parameterize the phase-space isolate the IR poles. perform the phase-space integration. results contain IR poles & finite terms.

Detailed calculation is given below but it is more interesting to understand the origin of the IR-poles.

* Virtual correction

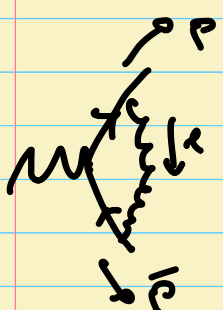
$$\bar{q}_0 \Gamma_0^\mu q_0 = Z_\Gamma Z_q \bar{q} \Gamma^\mu q$$ conserved current

$$Z_q = 1 - \frac{\alpha_s}{4\pi\epsilon} C_F$$, Feynman gauge

similar to QED case by with $\alpha \to \alpha_s$
$e_q^2 \to C_F$

$$Z_\Gamma = 1 + \frac{\alpha_s}{4\pi\epsilon} C_F$$ required by U(1) symmetry

can be obtained by



$$= \int [dl]\, \bar{u}_i\, i g_s \gamma^\alpha t^A_{ik}\, i\frac{\not{p}+\not{l}}{D_1} \gamma^\mu (-i) \frac{\bar{\not{p}}-\not{l}}{D_2}\, i g_s \gamma_\alpha t^A_{kj} v_j \frac{-i}{D_3}$$

$[dl] = \frac{d^d l}{(2\pi)^d}$. $D_1 = (p+l)^2 + i0^+$ $D_2 = (\bar{p}-l)^2 + i0^+$ $l^2 + i0^+$

$$= i g_{s_0}^2 (t^A t^A)_{ij} \int [dl]\, (D_1 D_2 D_3)^{-1} \cdot N^\mu(p, \bar{p}, l)$$

$$\bar{u}_i \gamma^\alpha (\not{p}+\not{l}) \gamma^\mu (\bar{\not{p}}-\not{l}) \gamma_\alpha v_j$$

Using Feyn. Parameterization, the denominator can be written as

$$(D_1 D_2 D_3)^{-1} = \Gamma(3)\int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(L^2+\alpha_2\alpha_3 S)^3}$$

$$\alpha_1 \ell^2 + \alpha_2 \ell^2 + 2\alpha_2 \ell\cdot p + \alpha_3 \ell^2 - 2\alpha_3 \ell\cdot\bar{p}$$

$$= \ell^2 + 2(\alpha_2 p - \alpha_3\bar{p})\cdot\ell + (\alpha_2 p - \alpha_3\bar{p})^2 - (\alpha_2 p - \alpha_3\bar{p})^2$$

$$= (\ell + \alpha_2 p - \alpha_3\bar{p})^2 + \alpha_2\alpha_3 S$$

$$\equiv L^2 + \alpha_2\alpha_3 S$$

The numerator N^μ can be simplified by throwing away linear terms in $L = \ell + \alpha_2 p - \alpha_3\bar{p}$

$$N^\mu = \bar{u}_i \gamma^\alpha (\not{p} + \not{L} - \alpha_2\not{p} + \alpha_3\bar{\not{p}})\gamma^\mu(\bar{\not{p}} - \not{L} + \alpha_2\not{p} - \alpha_3\bar{\not{p}})\gamma_\alpha v_j$$

$$= -\bar{u}_i\gamma^\alpha(\not{L} + \underset{=}{\bar{\alpha}_2}\not{p} + \alpha_3\bar{\not{p}})\gamma^\mu(\not{L} - \alpha_2\not{p} - \underset{=}{\bar{\alpha}_3}\bar{\not{p}})\gamma_\alpha v_j \qquad \bar{\alpha}_2 = 1-\alpha_2,\ \bar{\alpha}_3 = 1-\alpha_3$$

$$= -\bar{u}_i\gamma^\alpha\not{L}\gamma^\mu\not{L}\gamma_\alpha v_j + \text{linear terms in } L \text{ vanishing when integrating over } L$$

$$+ \bar{u}_i\gamma^\alpha(\bar{\alpha}_2\not{p} + \alpha_3\bar{\not{p}})\gamma^\mu(\alpha_2\not{p} + \bar{\alpha}_3\bar{\not{p}})\gamma_\alpha v_j$$

now we use

$$\gamma_\alpha \not a \gamma^\mu \not b \gamma^\alpha = -2 \not b \gamma^\mu \not a + 2\epsilon \not a \gamma^\mu \not b$$

to find

$$N^\mu \to 2(1-\epsilon)\, \bar u_i \not k \gamma^\mu \not k v_j \qquad \bar u \not p = \not{\bar p} v = 0$$

$$-2 \bar u_i (\alpha_2 \not p + \bar\alpha_3 \not{\bar p}) \gamma^\mu (\bar\alpha_2 \not p + \alpha_3 \not{\bar p}) v_j$$

$$+2\epsilon \bar u_i (\bar\alpha_2 \not p + \alpha_3 \not{\bar p}) \gamma^\mu (\alpha_2 \not p + \bar\alpha_3 \not{\bar p}) v_j$$

$$= 2(1-\epsilon)\, \bar u_i \not k \gamma^\alpha \not k v_j$$

$$- 2 \bar\alpha_2 \bar\alpha_3 \bar u_i \not{\bar p} \gamma^\mu \not p v_j$$

$$+ 2\epsilon \alpha_2 \alpha_3 \bar u_i \not{\bar p} \gamma^\mu \not p v_j$$

$$= 2(1-\epsilon)\, \bar u_i \not k \gamma^\alpha \not k v_j$$

$$- 2 \bar\alpha_2 \bar\alpha_3 (-s) \bar u_i \gamma^\mu v_j$$

$$+ 2\epsilon \alpha_2 \alpha_3 (-s) \bar u_i \gamma^\mu v_j$$

In the last step, we have used

$$\bar{u}\,\bar{\not{p}}\gamma^\mu \not{p} v = -\bar{u}\left(\cancel{\bar{\not{p}}\not{p}}\gamma^\mu - \cancel{2p^\mu \bar{\not{p}}}\right)v \quad \to 0 \quad \bar{\not{p}}v = 0$$

$$= -\bar{u}\left(2p\cdot\bar{p} - \not{p}\cancel{\bar{\not{p}}}\right)\gamma^\mu v = -s\,\bar{u}\gamma^\mu v$$

$$= 0$$

now we do the replacement within the loop integral

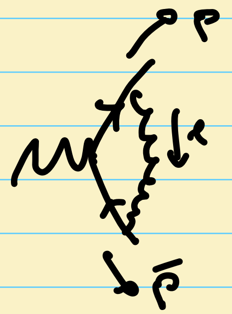
$$\not{L}\gamma^\mu\not{L} \to \frac{1}{d}L^2\gamma^\alpha\gamma^\mu\gamma_\alpha = -\frac{d-2}{d}L^2\gamma^\mu$$

which simplifies N^μ to

$$N^\mu \to -\frac{(d-2)^2}{d}L^2 \quad \bar{u}_i\gamma^\mu v_j$$
$$- 2\bar{\alpha}_2\bar{\alpha}_3(-s) \quad \bar{u}_i\gamma^\mu v_j$$
$$+ 2\epsilon\alpha_2\alpha_3(-s) \quad \bar{u}_i\gamma^\mu v_j$$

with $\bar{\alpha}_{2,3} = 1-\alpha_{2,3}$

Therefore we have



$$= i g_{s_0}^2 C_F \, \delta_{ij} \underbrace{\bar{u} \gamma^\mu v_j}_{\sim J^{\mu,(0)}} \Gamma(3)$$

$$\times \int d\alpha_1 d\alpha_2 d\alpha_3 \int [dL] \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(L^2+\alpha_2\alpha_3 s)^3}$$

$$\times \left\{ -\frac{(d-2)^2}{d} L^2 - 2\bar{\alpha}_2\bar{\alpha}_3(-s) + 2\epsilon\,\alpha_2\alpha_3(-s) \right\}$$

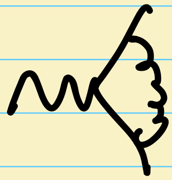
The loop Integration can be performed in a standard way, which gives

$$\int [dL] \frac{L^2}{(L^2+\alpha_2\alpha_3 s)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (\alpha_2\alpha_3)^{-\epsilon} (-s)^{-\epsilon}$$ ~ UV div.

$L \to \infty \Rightarrow \int \frac{d^4L \, L^2}{L^6} \Rightarrow$ logarithmic div. as $L \to \infty$

$$\int [dL] \frac{(-s)}{(L^2+\alpha_2\alpha_3)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} (\alpha_2\alpha_3)^{-1-\epsilon} (-s)^{-\epsilon}$$

$L \to \infty \Rightarrow \int \frac{d^4L}{L^6}$ ~ UV finite

$\Longrightarrow$  $= J^{\mu,(0)}\, g_{s,0}^2\, C_F \frac{(-1)}{(4\pi)^{d/2}} (-s)^{-\epsilon}\, \Gamma(1+\epsilon)$

$$\times \int_0^1 d\alpha_2 \int_0^{\bar{\alpha}_2} d\alpha_3 \left\{ -2(1-\epsilon)^2 (\alpha_2\alpha_3)^{-\epsilon} \frac{1}{\epsilon} \right.$$ UV

$$\left. + \left(2\bar{\alpha}_2\bar{\alpha}_3 - 2\epsilon\,\alpha_2\alpha_3\right) (\alpha_2\alpha_3)^{-1-\epsilon} \right\}$$

$$= J^{\mu,(0)} (-1) \frac{g_{s,0}^2}{(4\pi)^{d/2}} C_F (-s)^{-\epsilon}\, \Gamma(1+\epsilon)$$

$$\times \left\{ -\frac{1}{\epsilon_{UV}} - 1 + \frac{2}{\epsilon_{IR}^2} + \frac{4}{\epsilon_{IR}} + 9 - \frac{\pi^2}{3} \right\}$$

Similar to the QED case (see part 1.) we have

$$\frac{g_{s,0}^2}{(4\pi)^{d/2}} = \mu^{2\epsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha\, e^{\gamma_E \epsilon}$$

to find

 $$= J^{\mu,(0)} \frac{\alpha_s(\mu)}{4\pi} C_F \left(\frac{\mu^2}{-s}\right)^{\epsilon} \Gamma(1+\epsilon) e^{\gamma_E \epsilon}$$

$$\times \left\{ \frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}^2} - \frac{4}{\epsilon_{IR}} - 8 + \frac{\pi^2}{3} \right\}$$

$$= J^{\mu,(0)} \frac{\alpha_s(\mu)}{4\pi} C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}$$

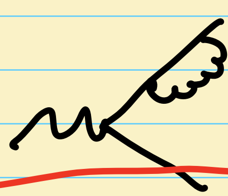
$$\times \left\{ \frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}^2} - \frac{4}{\epsilon_{IR}} - \frac{2i\pi}{\epsilon_{IR}} - 8 + \frac{7\pi^2}{6} - 3i\pi \right\}$$

The result confirms $Z_J = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F$.

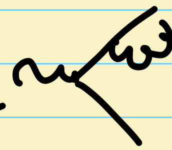
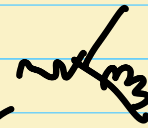
and also gives the IR divergence & finite terms

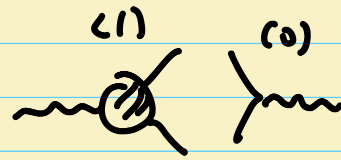
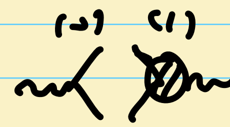
additional virtual corrections are scaleless

and hence vanishes in dim-reg.

 $$= 0 = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$$

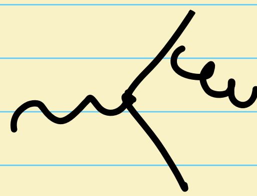
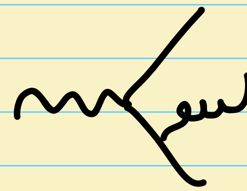
add up everything

 $+\ \frac{1}{2}$  $+\ \frac{1}{2}$  $-\ \delta Z_\psi - \delta Z_\Gamma$

$\Rightarrow \frac{\alpha_s}{2\pi}\sigma^{(1)}_{virt} =$  $+$ 

$$= \sigma^{(0)}\frac{\alpha_s}{2\pi}C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\times\left\{-\frac{2}{\epsilon_{IR}^2}-\frac{3}{\epsilon_{IR}}-8+\frac{7}{6}\pi^2\right\}$$

** Real Correction ✓

$\int d\Phi_3 \frac{e_q^2}{d-1}\frac{1}{q^2}\Bigg|$  $+$  $\Bigg|^2$

$=$

↳ $3 \to d-1$ in dim reg. since $g_{\mu\nu}g^{\mu\nu} = d$

$$= C_F g_s^2 \underbrace{\left[-\frac{4(1-\epsilon)}{d-1}\right]}_{\to \text{LO}} \frac{2}{q^2}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

where

$$y_1 = \frac{s_{12}}{q^2}, \quad y_2 = \frac{s_{13}}{q^2}, \quad y_3 = \frac{s_{23}}{q^2}$$

and $s_{ij} = 2 p_i \cdot p_j$.

$1 \to q, \quad 2 \to \bar{q}, \quad 3 \to g.$

now $X = q.\bar{q}.g$

$$d\Phi_3 = \frac{1}{(2\pi)^{2d-3}} \frac{1}{2^{d+1}} s^{d-3}$$

$$d\Omega_{d-1}\, d\Omega_{d-2}\, (y_1 y_2 y_3)^{-\epsilon}$$

$$\int_0^1 dy_1\, dy_2\, dy_3\, \delta(1-y_1-y_2-y_3)$$

$$\boxed{= \Phi_2 \cdot \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(1-\epsilon)} (q^2)^{1-\epsilon} \times \int_0^1 dy_1\, dy_2\, dy_3\, \delta(1-y_1-y_2-y_3) (y_1 y_2 y_3)^{-\epsilon}}$$

Integrate over the phase space we find the real correction be

$$\int d\Phi_3 \frac{1}{d-1}\frac{1}{q^2}\left| \mathcal{M}_{\gamma^*\to q\bar{q}g} + \mathcal{M}_{\gamma^*\to q\bar{q}g} \right|^2$$

$$= \Phi_2 \left[-\frac{4(1-\epsilon)}{d-1}\right] \frac{g_s^2\, 2C_F}{(4\pi)^{\epsilon/2}} \frac{1}{\Gamma(1-\epsilon)} (q^2)^{-\epsilon}$$

$$\int_0^1 dy_2 \int_0^{\bar{y}_2} dy_3$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

Poles from y_2 & $y_3 \to 0$ double pole

or $y_2 \to 0$ or $y_3 \to 0$ single pole

$$= \sigma^{(0)} \frac{\alpha_s(\mu)}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{q^2}{\mu^2}\right)^{-\epsilon}$$

$$\times \left\{ \frac{2}{\epsilon_{IR}^2} + \frac{3}{\epsilon_{IR}} + \frac{19}{2} - \pi^2 \right\}$$

$$= \sigma^{(0)} \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{q^2}{\mu^2}\right)^{-\epsilon}$$

$$\times \left\{ \frac{2}{\epsilon_{IR}^2} + \frac{3}{\epsilon_{IR}} + \frac{19}{2} - \frac{7}{6}\pi^2 \right\}$$

Deriviction of $d\Phi_3$

$$d\Phi_3 = \frac{1}{(2\pi)^{d-1}}\frac{d^{d-1}p_1}{2p_1}\ \frac{1}{(2\pi)^{d-1}}\frac{d^{d-1}p_2}{2p_2}$$

$$\times \frac{d^d p_3}{(2\pi)^{d-1}} \times \delta(p_3^2)\ (2\pi)^d \delta^{(d)}(q-p_1-p_2-p_3)$$

$$= \left[\frac{1}{(2\pi)^{d-1}}\right]^2 \frac{\pi}{2}\ p_1^{d-3}\, dp_1\, d\Omega_1\ p_2^{d-3}\, dp_2\, d\Omega_2$$

$$\times\ \delta\left((q-p_1-p_2)^2\right)$$

$$\Rightarrow q^2 + 2p_1\cdot p_2 - 2p_1\cdot q - 2p_2\cdot q$$

in the center of mass frame
we define

$$x_i = \frac{2p_i}{q} \Rightarrow \sum_i x_i = 2$$

Therefore. the phase space can be written as

$$d\Phi_3 =$$

$$= \left[\frac{1}{(2\pi)^{d-1}}\right]^2 \frac{\pi}{2} \frac{1}{4} \left(\frac{1}{2}\right)^{2d-6} (q^2)^{d-3}$$

$$(x_1 x_2)^{d-3} \, dx_1 \, dx_2 \, d\Omega_1 \, d\Omega_2$$

$$\times \delta\left(1 - x_1 - x_2 + x_1 x_2 \xi_{12}\right)$$

Here $\xi_{12} \equiv \frac{1}{2}(1-\cos\theta_{12})$.

The solid angle integration is given by

$$d\Omega_2 = \sin^{d-3}\theta_{12} \, d\theta_{12} \, d\Omega_{d-2}$$

$$= (1-\cos^2\theta_{12})^{\frac{d-4}{2}} \, d\cos\theta_{12} \, d\Omega_{d-3}$$

$$= 2^{d-3} \xi_{12}^{\frac{d-4}{2}} (1-\xi_{12})^{\frac{d-4}{2}} \, d\xi_{12} \, d\Omega_{d-3}$$

$$\Rightarrow d\Phi_3 = \left[\frac{1}{(2\pi)^{d-1}}\right]^2 \frac{\pi}{2} \frac{1}{4} \left(\frac{1}{2}\right)^{d-3} (q^2)^{d-3}$$

$$dx_1\, dx_2\, d\Omega_1\, d\Omega_{d-2}$$

$$(x_1 + x_2 - 1)^{\frac{d-4}{2}} \left[(1-x_1)(1-x_2)\right]^{\frac{d-4}{2}}$$

Now we write x_1 & x_2 interms of Lorentz invariant quantities.

$$2p_1 \cdot q = 2E_1 q = x_1 q^2 = 2p_1\cdot p_2 + 2p_1 \cdot p_3 = (y_1 + y_2) q^2$$

$$2p_2 \cdot q = 2E_2 q = x_2 q^2 = 2p_1 \cdot p_2 + 2p_2 \cdot p_3 = (y_1 + y_3) q^2$$

$$\Rightarrow \quad x_1 = y_1 + y_2 = 1 - y_3$$

$$x_2 = y_1 + y_3 = 1 - y_2$$

Therefore

$$d\Phi_3 = \frac{1}{(2\pi)^{2d-3}} \left(\frac{1}{2}\right)^{d+1} (q^2)^{d-3}$$

$$d\Omega_{d-2}\, d\Omega_{d-1}$$

$$dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{\frac{d-4}{2}}\, \delta(1-y_1-y_2-y_3)$$

where Ω is the solid angle

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}$$

- Soft & Collinear limit

Now we aim to understand the origin of the IR poles. This can be achieved by examing either the virtual (Landau Eqn) and real correction. the latter is more intuitive.

IR poles could occur when the matrix element has singularities

In our case

$$|\mathcal{M} + \mathcal{M}|^2$$

$$\sim \frac{2}{y_2 y_3} + \frac{-2+(1-\epsilon)y_3}{y_2} + \frac{-2+(1-\epsilon)y_2}{y_3} - 2\epsilon$$

it is when

i) both y_2 & $y_3 \to 0$

ii) only $y_2 \to 0$

iii) only $y_3 \to 0$

We notice that

$$y_2 \sim P_1 \cdot P_3 = E_1 E_3 (1 - \cos\theta_{13})$$

$$y_3 \sim P_2 \cdot P_3 = E_2 E_3 (1 - \cos\theta_{23})$$

$\Rightarrow$ i) $E_3 \approx 0$, gluon is soft

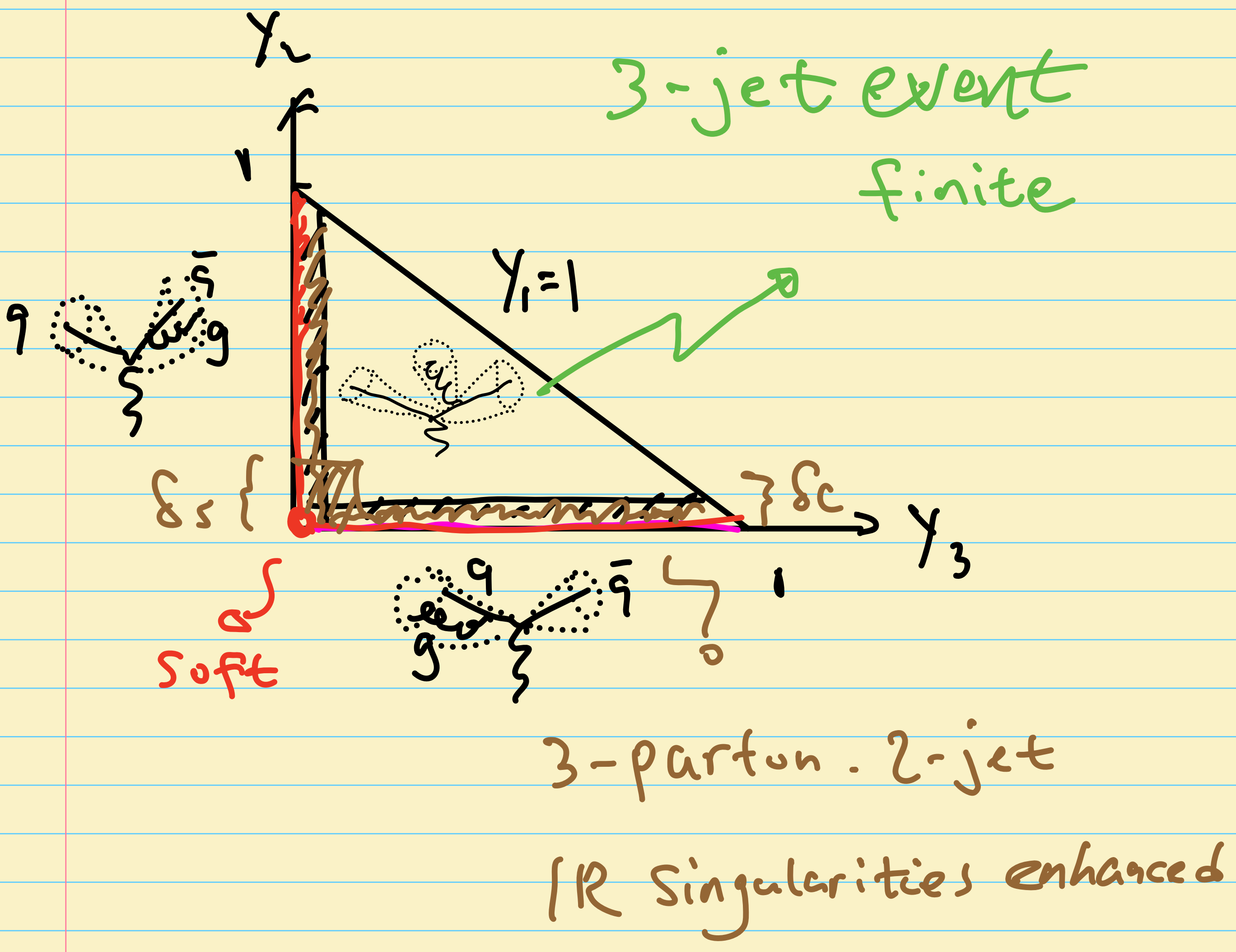
ii) $\theta_{13} \simeq 0$ $\quad g \parallel q$, collinear

iii) $\theta_{23} \approx 0$ $\quad g \parallel \bar{q}$, collinear

degeneracy leads to the IR divergence, soft & collinear

quarks do not lead to IR poles!!

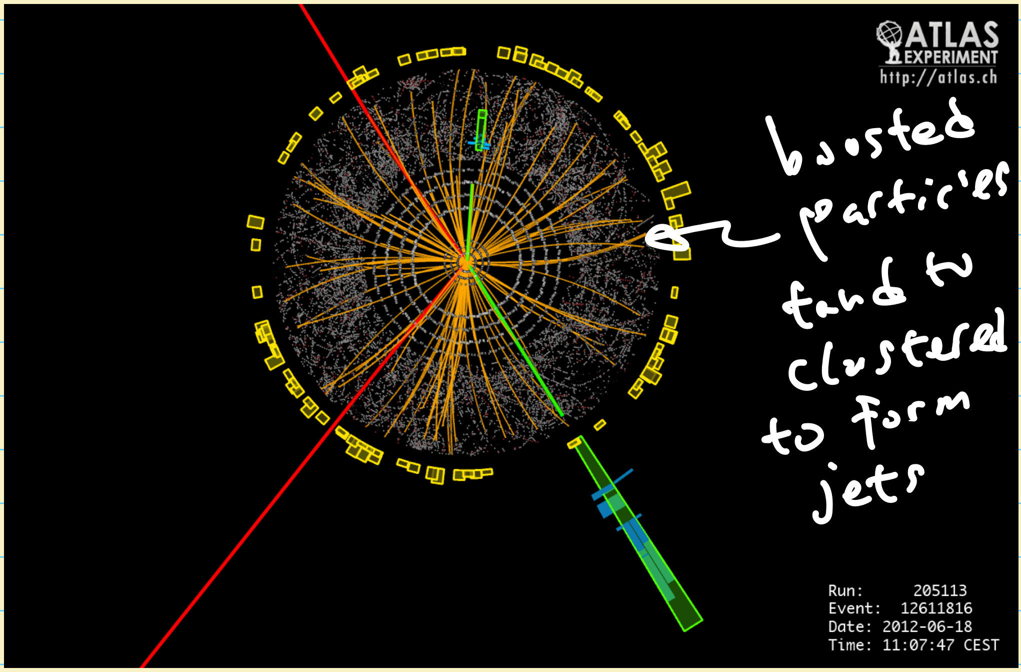
phase - space



an event is a point in the phase space. with weight $|M|^2$

So 2-jet event (3-parton) is enhanced by IR singularities compared with 3-jet event (3-parton).

the same reason explains the jet-like events at the LHC. (collinear & soft singularities enhanced configuration)



In the soft & collinear limit, the matrix is dramatically simplified

i) soft: $|M^{(0)}|^2 \, 2 g_s^2 C_F \frac{1}{q^2} \frac{2}{y_2 y_3}$

$$= |M^{(0)}|^2 \, 2 g_s^2 C_F \underbrace{\frac{p_1 \cdot p_3}{p_1 \cdot p_3 \; p_2 \cdot p_3}}_{\text{eikonal factor}}$$

ii) collinear: $|M^{(0)}|^2$

$$g_s^2 C_F \frac{2}{q^2} \frac{1}{y_3} \left\{ \frac{2}{y_2} - 2 + (1-\epsilon) y_2 \right\}$$

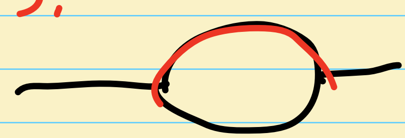
$$= |M^{(0)}|^2 \, g_s^2 C_F \frac{2}{2 p_2 \cdot p_3} \underbrace{\left\{ \frac{1 + (1-y_2)^2}{y_2} - \epsilon y_2 \right\}}_{\text{splitting kernel}}$$

actually these forms are Universal (process independent)

And we try to understand the general IR feature now.

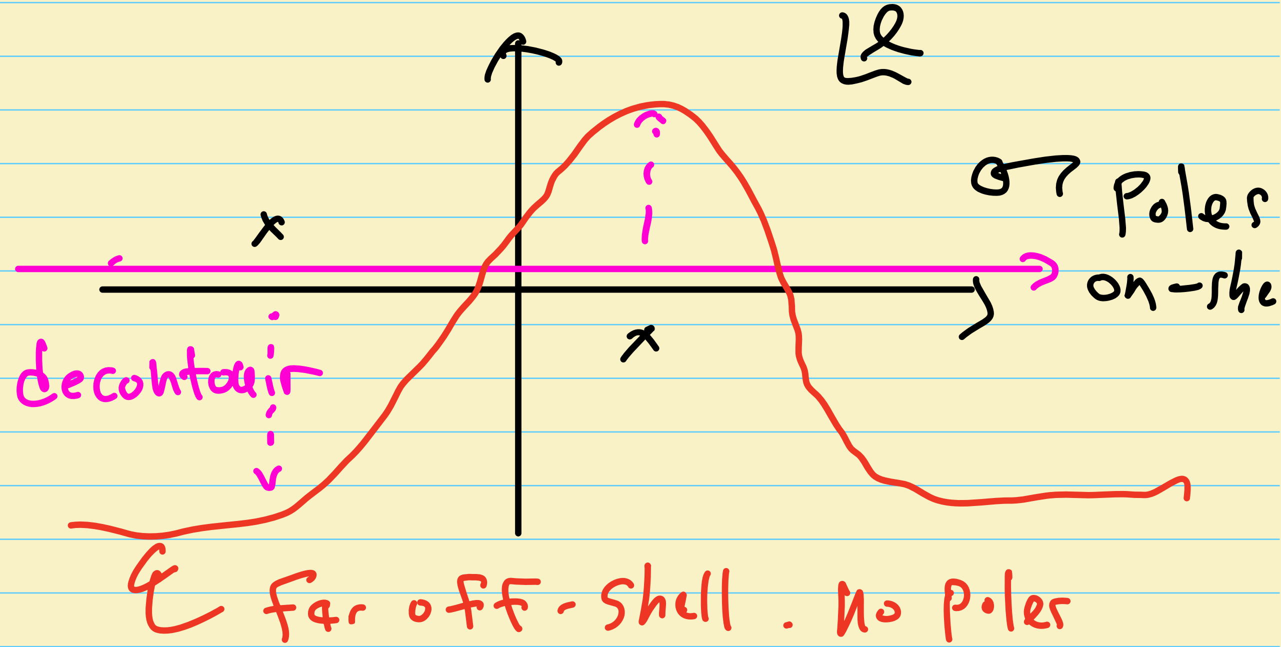
* Sketch of Visit.

e.g. $D_1 = 0$, on shell

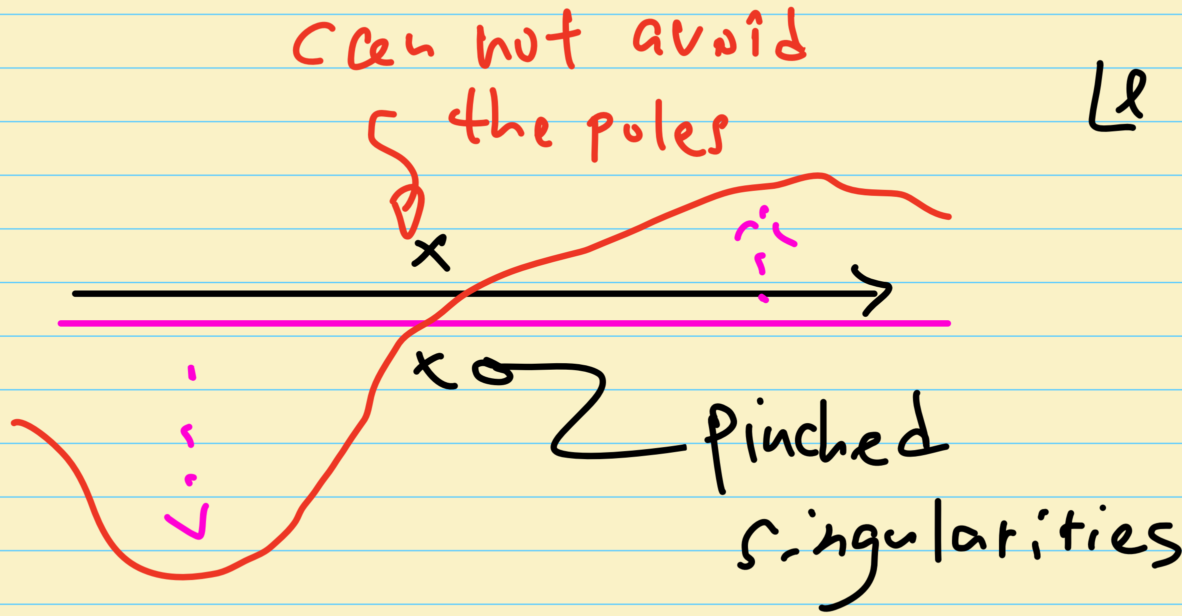


$$\propto \int d\ell \, \frac{1}{D_1 D_2}$$

singularities may arise when $D_i \to 0$ (poles, on-shell) but it is not a sufficient condition



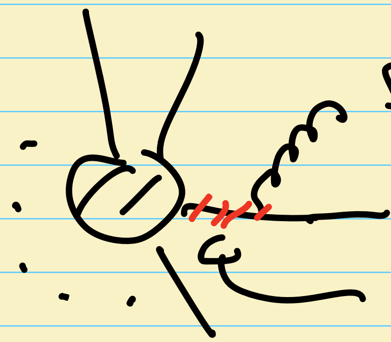
but if the singularities are pinched or end points, then there will be a pole associated



end point

more comprehensive analysis using Landau eqn. $\Rightarrow$ soft & collinear + potential mode

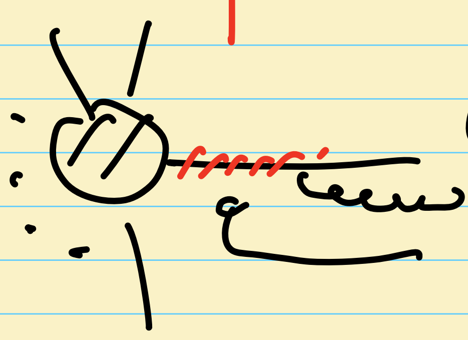
* More intuitively by real



soft. $E_g \to 0$

$D = 2p\cdot g \to 0$

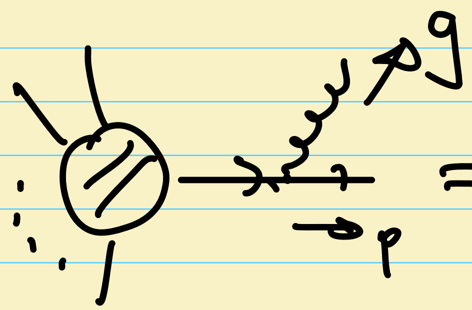
two ways to make the internal line on-shell



Collinear. $\theta \to 0$

$D = 2p\cdot g \to 0$

$\Longrightarrow$ eikonal approximation



$$= \bar{u}\, i g_s t^A \not{\epsilon}\, \frac{i(\not{p}+\not{q})}{(p+q)^2+i0^+} A$$

$$\simeq \bar{u}\, i g_s t^A \not{\epsilon}\, \frac{i\not{p}}{2p\cdot q+i0^+} A$$

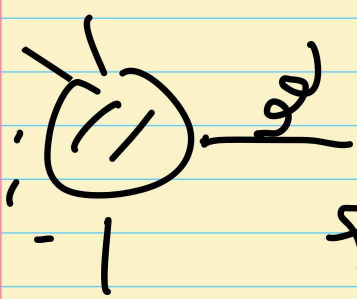
$\not{\epsilon}\not{p} = 2p\cdot\epsilon - \not{p}\not{\epsilon}$

$\bar{u}\not{p} = 0$

$$= \bar{u}\,(-g_s t^A)\, \frac{2p^\mu \epsilon_\mu}{2p\cdot q+i0^+} A$$

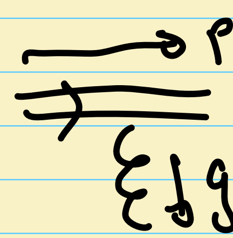
$$= -g_s t^A \frac{p\cdot\epsilon}{p\cdot q+i0^+} \mathcal{M}_0$$

Hence, we have



$\xrightarrow[\text{factorize}]{q^0 \to 0}$

$\otimes$

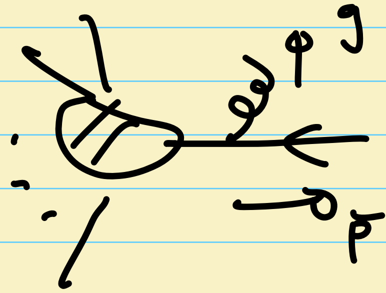


$$\mathcal{M}_0 \cdot \boxed{-g_s t^A \frac{p^\mu}{p\cdot q+i0^+}}$$

eikonal factor

spin independent.

similarly

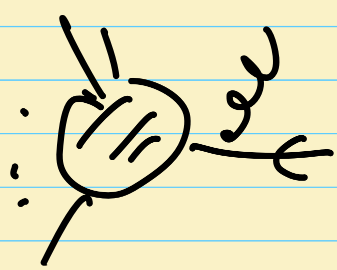


$$\mathcal{A}\,\frac{-i(\bar{\not p}+\not q)}{2\bar p\cdot q+i0^+}\, i g_s t^A \not\epsilon\, v$$

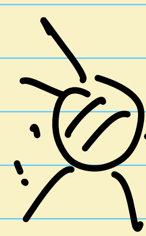
$$= \mathcal{A}\,\frac{\bar{\not p}\,\not\epsilon}{2\bar p\cdot q+i0^+}\, g_s t^A v$$

$$= +g_s t^A\,\frac{\bar p\cdot\epsilon}{\bar p\cdot q+i0^+}\,\mathcal{M}_0$$

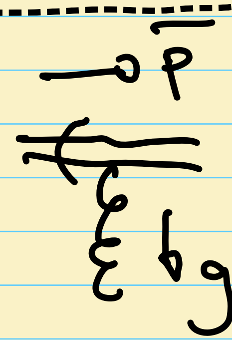
which gives



$$\overset{q\to 0}{=}$$

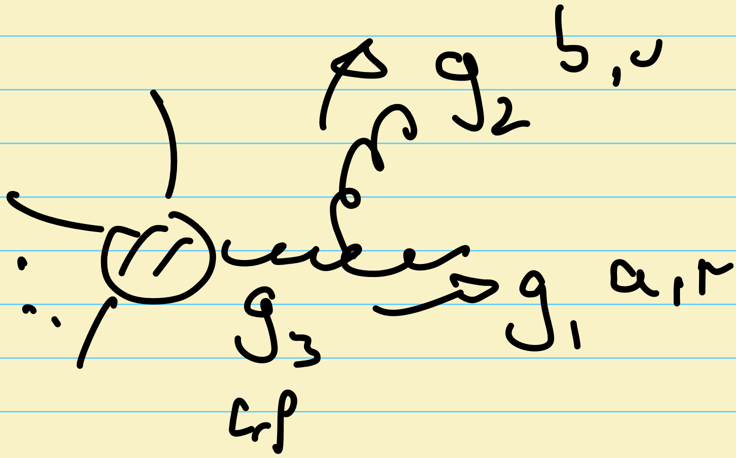


$\otimes$



$\mathcal{M}_0$

$$+g_s t^A\,\frac{\bar p^\mu}{\bar p\cdot q+i0^+}$$



$, \; g_2 \to 0$

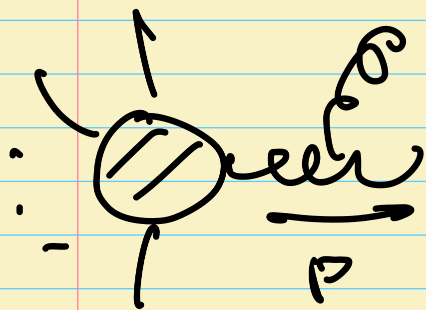
$g_1 = g_3$

$$= g_s f^{abc} \frac{-i}{2 g_1 \cdot g_1 + i0^+} \Big[\epsilon_1 \cdot \epsilon_2 (-g_1 + g_2)^\rho + \epsilon_2^\rho (-g_2 - g_3) \cdot \epsilon_1 + \epsilon_1^\rho (g_3 + g_1) \cdot \epsilon_2 \Big] A_\rho$$

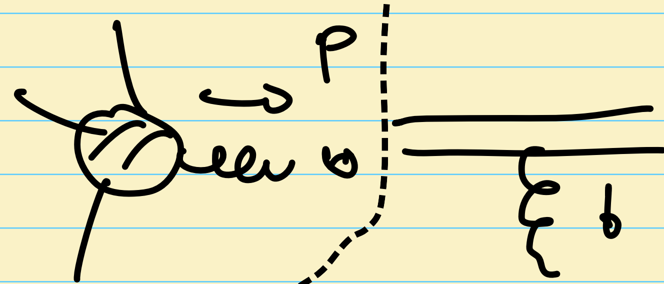
$$\simeq g_s f^{abc} \frac{-i}{2 g_1 \cdot g_2 + i0^+} \Big[\epsilon_1 \cdot \epsilon_2 \cancelto{0}{(-g_2^\rho)} A_\rho + \epsilon_2^\rho \cancelto{0}{(-g_2)} \cdot \epsilon_1 A_\rho + \epsilon_1^\rho \, 2 g_1 \cdot \epsilon_2 A_\rho \Big]$$

Wald identity

Therefore, one finds

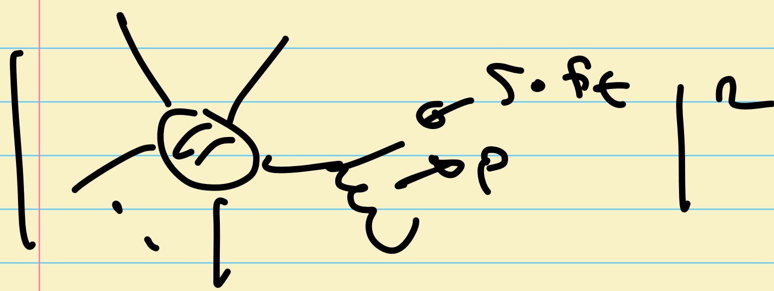


$$\simeq$$



$$\mu \circ \quad \cdot\; -i g_s f^{abc} \frac{p^\mu}{p\cdot q + i\epsilon}$$

What happens to soft quarks?

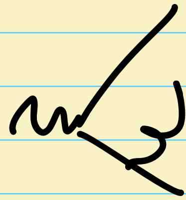


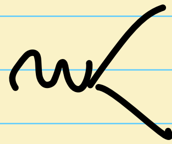
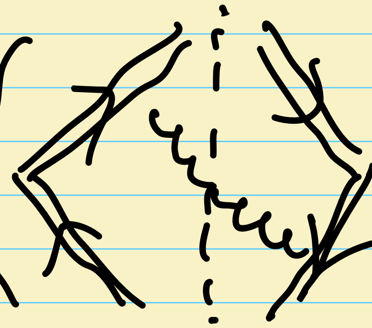
$\propto \not{p} \times \ldots\ldots$

suppressed by $\not{p} \to 0$ in the numerator

does not lead to leading singularities

One can then get to leading soft contribution using eikonal approximation w/o going through the full calculation for instance

$$\Big|$$  $$+$$  $$\Big|^2$$

$$= \Big|$$  $$\Big|^2 \cdot \Big($$  $$+ \text{c.c.}\Big)$$

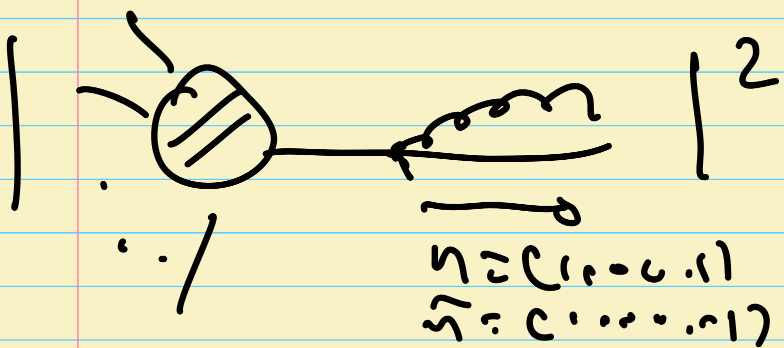
$$= |M_0|^2 \cdot \left(-g_s^2 t^A t^A \frac{p\cdot\bar{p}}{-p\cdot g\,\bar{p}\cdot g} \cdot 2\right)$$

$$= |M_0|^2 \; g_s^2 C_F \; 2 \, \frac{p\cdot\bar{p}}{p\cdot g\,\bar{p}\cdot g}$$

$$= |M_0|^2 g_s^2 C_F \frac{4}{q^2} \frac{1}{y_2 y_3}$$

reproduces the soft limit of the full process

⇒ collinear factorization

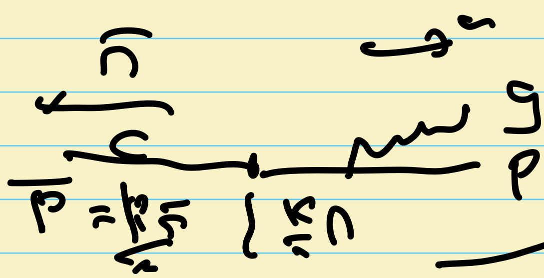


physical gauge to avoid interference

$$p^\mu = z k^\mu + p_\perp^\mu - \frac{p_\perp^2}{2z\,\bar{n}\cdot k}\bar{n}^\mu$$

$$q^\mu = \bar{z} k^\mu - p_\perp^\mu - \frac{p_\perp^2}{2\bar{z}\,\bar{n}\cdot k}\bar{n}^\mu$$

$$2p\cdot q = \frac{-p_\perp^2}{z\bar{z}} \qquad p\cdot p = q\cdot q = 0$$



$\bar{z}\cdot k^\mu \bar{n}_\mu$

$\bar{P} = k\cdot\bar{n}$

$$\boxed{\begin{aligned} 2\bar{P}\cdot q &= y_3 q^2 = \bar{z} q^2 \\ 2P\cdot q &= y_2 q^2 \end{aligned}}$$

$$\bar{u}\, i g_s t^a \not\epsilon \frac{i(\not p+\not g)}{2p\cdot g+i0^+} A A^\dagger \frac{-i(\not p+\not g)}{2p\cdot g-i0^+} \not\epsilon(-i g_s t^a) u.$$

$$= g_s^2 C_F \left(\frac{1}{2p\cdot g}\right)^2 \mathrm{Tr}\left[\not p \not\epsilon (\not p+\not g) A A^\dagger (\not p+\not g)\not\epsilon\right]$$

$$\sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu = -g^{\mu\nu} + \frac{\bar{n}^\mu g^\nu + \bar{n}^\nu g^\mu}{\bar{n}\cdot g}$$ axial gauge

$$\Rightarrow \mathrm{Tr}[\cdots]$$

$$= -\mathrm{Tr}\left[\not p \gamma^\mu (\not p+\not g) A A^\dagger (\not p+\not g)\gamma_\mu\right] \quad ①$$

$$+ \frac{1}{\bar{n}\cdot g}\mathrm{Tr}\left[\not p \not{\bar{n}} (\not p+\not g) A A^\dagger (\not p+\not g)\not g\right] \quad ②$$

$$+ \frac{1}{\bar{n}g}\mathrm{Tr}\left[\not p \not g (\not p+\not g) A A^\dagger (\not p+\not g)\not{\bar{n}}\right] \quad ③$$

② = ③

to get $\frac{1}{2p\cdot g}$ pole. we need keep $(2p\cdot g)$ term in the numerator

$$① = 2(1-\epsilon)\,\mathrm{Tr}\left[\cancel{p}(\cancel{p}+\cancel{g})AA^\dagger(\cancel{p}+\cancel{g})\right]$$

$$= 2(1-\epsilon)\,\mathrm{Tr}\left[\cancel{p}\cancel{g}AA^\dagger\cancel{g}\right]$$

$$= 2(1-\epsilon)\,2p\cdot g\,\mathrm{Tr}\left[AA^\dagger\cancel{g}\right]$$

$$= 2(1-\epsilon)\,2p\cdot g\,\bar{z}\,\underbrace{\mathrm{Tr}\left[AA^\dagger\cancel{k}\right]}_{|M_0|^2}$$

$$= 2(1-\epsilon)\,2p\cdot g\,\bar{z}\,|M_0|^2$$

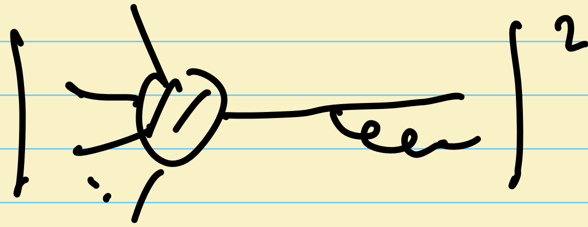
$$② = \frac{1}{\bar{n}\cdot g}\mathrm{Tr}\left[\cancel{p}\cancel{\bar{n}}(\cancel{p}+\cancel{g})AA^\dagger(\cancel{p}+\cancel{g})\cancel{g}\right]$$

$$= \frac{1}{\bar{n}\cdot g}\mathrm{Tr}\left[\cancel{p}\cancel{\bar{n}}(\cancel{p}+\cancel{g})AA^\dagger\cancel{p}\cancel{g}\right]$$

$$= (2p\cdot g)\frac{1}{\bar{n}\cdot g}\mathrm{Tr}\left[\cancel{p}\cancel{\bar{n}}(\cancel{p}+\cancel{g})AA^\dagger\right]$$

$$= (2p\cdot g)\,\frac{2\bar{n}\cdot p}{\bar{n}\cdot g}\,\mathrm{Tr}\left[\cancel{k}A^\dagger A\right]$$

$$= (2p\cdot g)\,\frac{2z}{1-z}\,|M_0|^2$$

$\Rightarrow$ 

$$= g_s^2\, C_F\, \frac{1}{2p\cdot q} \left\{ \frac{4z}{\bar z} + 2\bar z - 2\epsilon \bar z \right\} |M_0|^2$$

$$= g_s^2\, C_F\, \frac{1}{p\cdot q} \underbrace{\left\{ \frac{1+z^2}{1-z} - \epsilon(1-z) \right\}}_{P_{q\to qg}(z,\epsilon)} |M_0|^2$$

$$P^{(0)}_{q\to qg} = C_F\, \delta_{ss'} \left(\frac{1+z^2}{1-z} - \epsilon(1-z) \right)$$

$$P^{(0)}_{q\to gq} = C_F\, \delta_{ss'} \left(\frac{1+(1-z)^2}{z} - \epsilon z \right)$$

$$P^{(0)\mu\nu}_{g\to q\bar q} = T_R \left(-g^{\mu\nu} + 4z(1-z)\frac{p_\perp^\mu p_\perp^\nu}{p_\perp^2} \right)$$

$$P^{(0)\mu\nu}_{g\to gg} = 2C_A \left(-g^{\mu\nu} \left[\frac{z}{\bar z} + \frac{\bar z}{z} \right) - 2(1-\epsilon) z\bar z \frac{p_\perp^\mu p_\perp^\nu}{p_\perp^2} \right.$$

$$\boxed{\bar{z} = y_J , \quad 2p\cdot q \sim Q^2 y_2}$$

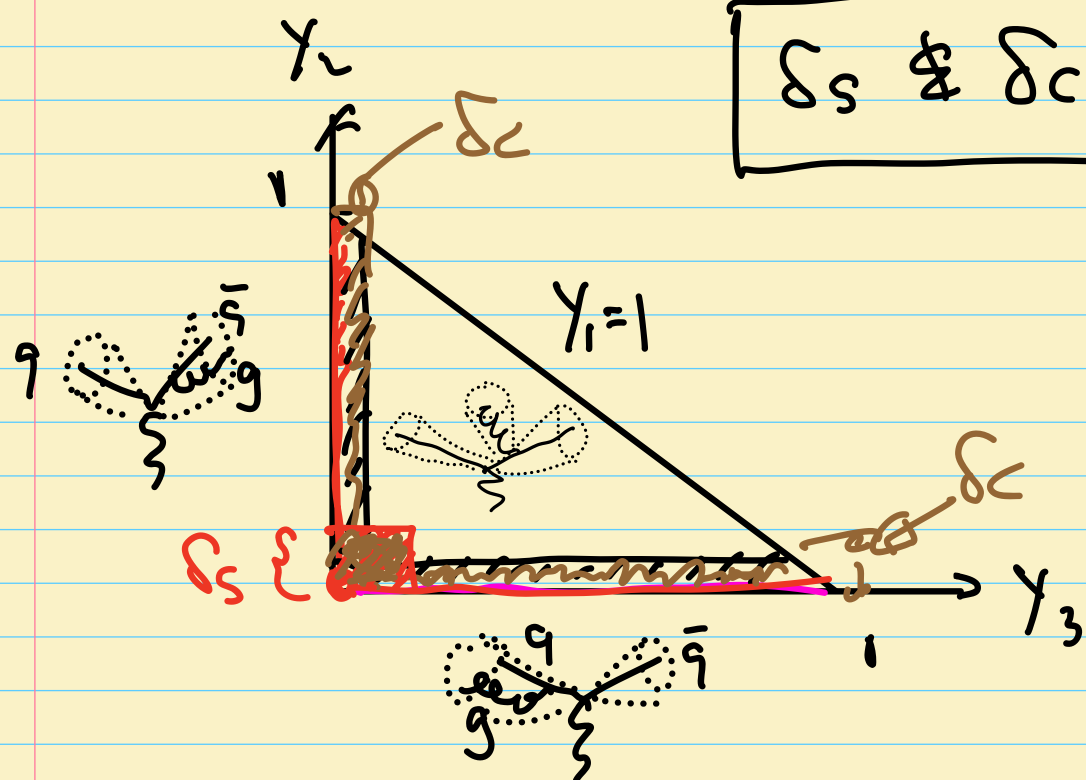
In high energy processes. soft & collinear radiation are usually dominated to form jets.

lets go more exclusive to understand the jet cross section.

— Jets & IR safe

di-jet event constrain to the shaded area

$$\boxed{\delta_s \ \& \ \delta_c << 1}$$



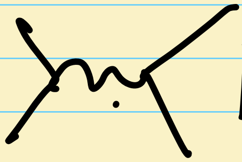
soft: $0 < y_2 < \delta_s, \ 0 < y_3 < \delta_s$

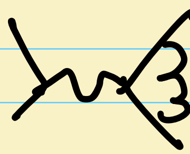
collinear: $0 < y_2 < \delta_c, \ \delta_s < y_3 < 1$

collinear: $0 < y_3 < \delta_c \quad \delta_s < y_2 < 1$

In this region we have

a) 2-parton contribution

$\Big|$  $\Big|^2 = \sigma^{(0)}$

$\Big|$  $\Big|^2 = \frac{\alpha_s}{2\pi}\sigma^{(1)}_{virt}$

$$= \sigma^{(0)}\frac{\alpha_s}{2\pi}C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\left\{-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{7}{6}\pi^2\right\}$$

b) 3-Parton contribution

dominated by soft & Collinear

Instead of using full $|M|^2$

we use soft & coll. approximation

Soft contribution $y_1 \simeq 1$

$$\sigma^{(1)}_{soft} = \sigma^{(0)}$$

$$\times \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(1-\epsilon)} (q^2)^{1-\epsilon} \; 2 g_s^2 C_F \frac{1}{q^2}$$

$$\times \int_0^{\delta_s} dy_2 \int_0^{\delta_s} dy_3 \, y_2^{-\epsilon} y_3^{-\epsilon} \frac{2}{y_2 y_3}$$

$$= \sigma^{(0)} \frac{g_s^2}{(4\pi)^{d/2}} \frac{1}{\Gamma(1-\epsilon)} (s)^{-\epsilon} C_F \frac{4}{\epsilon^2} \delta_s^{-2\epsilon}$$

$$= \sigma^{(0)} \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{s}\right)^{\epsilon} \frac{2}{\epsilon^2} \delta_s^{-2\epsilon}$$

$$= \sigma^{(0)} \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}$$

$$\times \left\{ \underbrace{\frac{2}{\epsilon^2}}_{\text{double pole}} - \frac{4 \log \delta_s}{\epsilon} - \frac{\pi^2}{6} + 4 \log^2 \delta_s \right\}$$

Collinear contribution $y_1 \simeq 1$

$$\sigma^{(1)}_{coll.} = 2\sigma^{(0)}$$

$$\times \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(1-\epsilon)} (q^2)^{1-\epsilon} \, 2 g_s^2 C_F \frac{1}{q^2}$$

$$\times \int_0^{\delta_c} dy_3 \, y_3^{-1-\epsilon}$$

$$\int_{\delta_s}^{1} y_2^{-\epsilon} \left\{ \frac{2}{y_2} - 2 + (1-\epsilon) y_2 \right\}$$

$$= \sigma^{(0)} \quad \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{s}\right)^{\epsilon}$$

$$-\frac{1}{\epsilon} \delta_c^{-\epsilon} \left\{ -\frac{3}{2} - 2\log\delta_s - \frac{9}{4}\epsilon + \epsilon \log^2 \delta_s \right\}$$

$$= \sigma^{(0)} \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}$$

$$\times \left\{ \frac{3}{\epsilon} + \frac{4\log\delta_s}{\epsilon} + \frac{9}{2} - 3\log\delta_c - 4\log\delta_s \log\delta_c \right\}$$

↳ single coll.

$$\Rightarrow \sigma_{2jet} = \sigma^{(0)} \Big\{ 1 + \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{s}\right)^{\epsilon} \times$$

$$\Big[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2$$

$$+ \quad + \frac{2}{\epsilon^2} - \frac{4\log\delta_s}{\epsilon} - \frac{\pi^2}{6} + 4\log^2\delta_s$$

$$+ \frac{3}{\epsilon} + \frac{4\log\delta_s}{\epsilon} + 9/2 - 3\log\delta_c$$

$$- 4\log\delta_c \log\delta_s \Big] \Big\}$$

$$= \sigma^{(0)} \Big\{ 1 + \frac{\alpha_s}{2\pi} C_F$$

$$\times \Big[-\frac{7}{2} + \pi^2 + 4\log^2\delta_s - 4\log\delta_s\log\delta_c$$

$$- 3\log\delta_c \Big] \Big\}$$

* poles cancel
* possible large logs → resumation

$$\# \; \sigma_{3\text{-jet}} = \sigma - \sigma_{2\text{jet}}$$

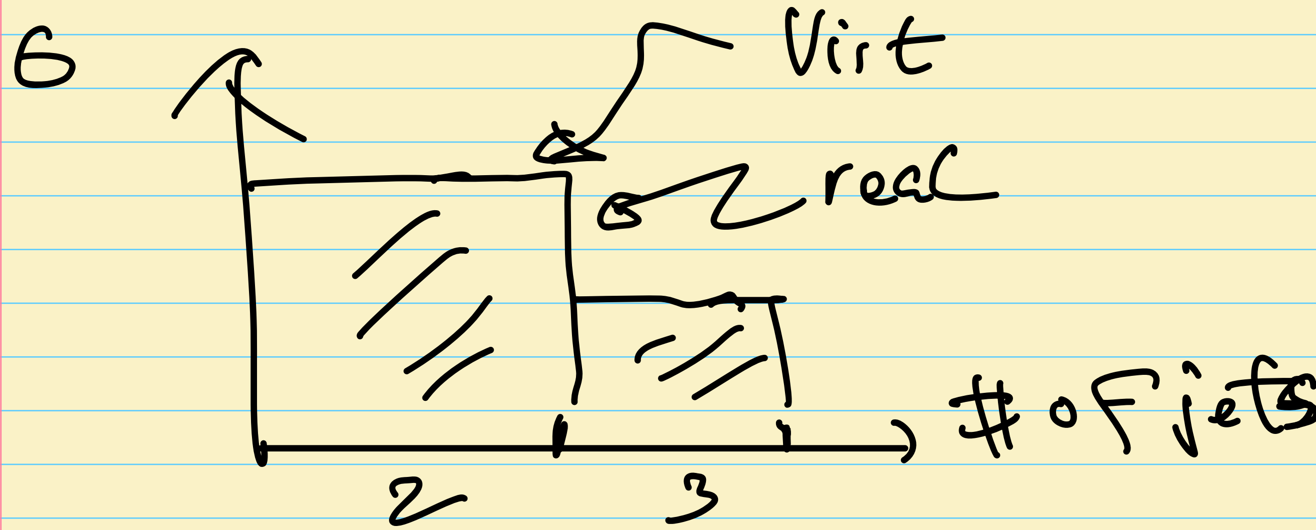
$$= \sigma^{(0)} \frac{\alpha_s}{2\pi} C_F$$

$$\times \left[5 - \pi^2 - 4 \log^2 \delta_s + 4 \log \delta_s \log \delta_c + 3 \log \delta_c \right]$$

both depends on cut-off parameters sum does not

all poles cancel again.

since virtual & real in the same bin



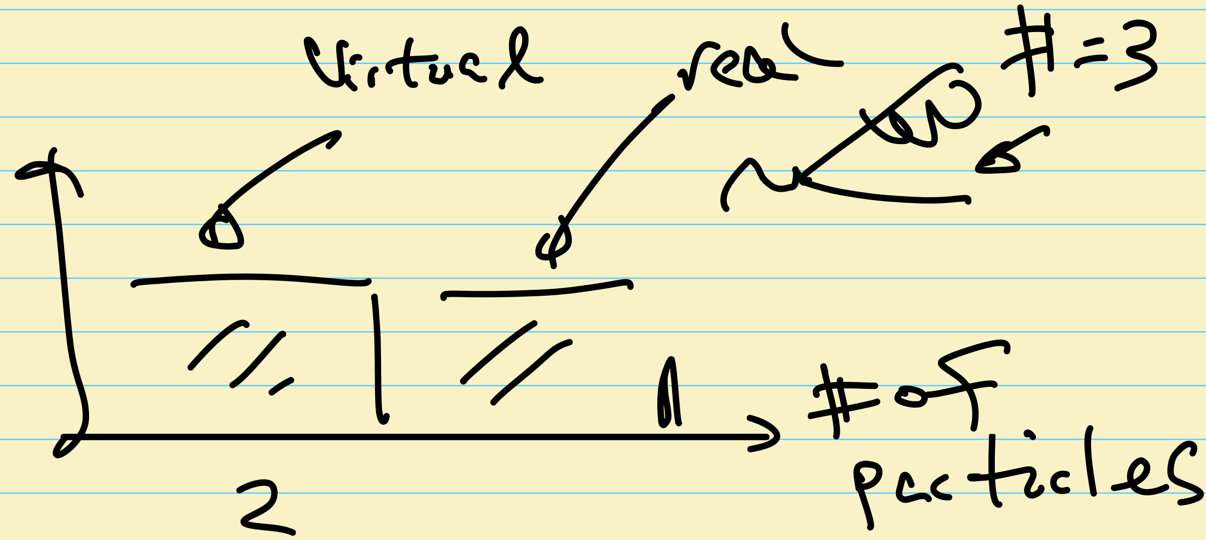
IR safety.

$$G_N(\cdots p_i \cdots p_j \cdots) \overset{p_i // p_j}{=} G_{N-1}(\cdots p_i + p_j \cdots)$$

$$G_N(\cdots p_i \cdots) \overset{p_i = 0}{=} G_{N-1}(\cdots, \cdots)$$

IR unsafe quantities:

particle #.



$$G_N = 3 \neq G_{N-1} = 2$$

usually we stick to IR safe quantities as required by fixed-order calculation.

large logs : $\alpha_s L^2$. $\alpha_s L \sim 1$

the Fixed order calculation is no longer valid

$$G \sim 1 + \alpha_s L^2 + \alpha_s L + C$$
$$+ \alpha_s^2 L^4 + \alpha_s^2 L^3 + \alpha_s^2 L^2 + \alpha_s^2 L + C$$
$$+ \cdots$$

if $\alpha_s L^2 \sim 1$ or $\alpha_s L \sim 1$

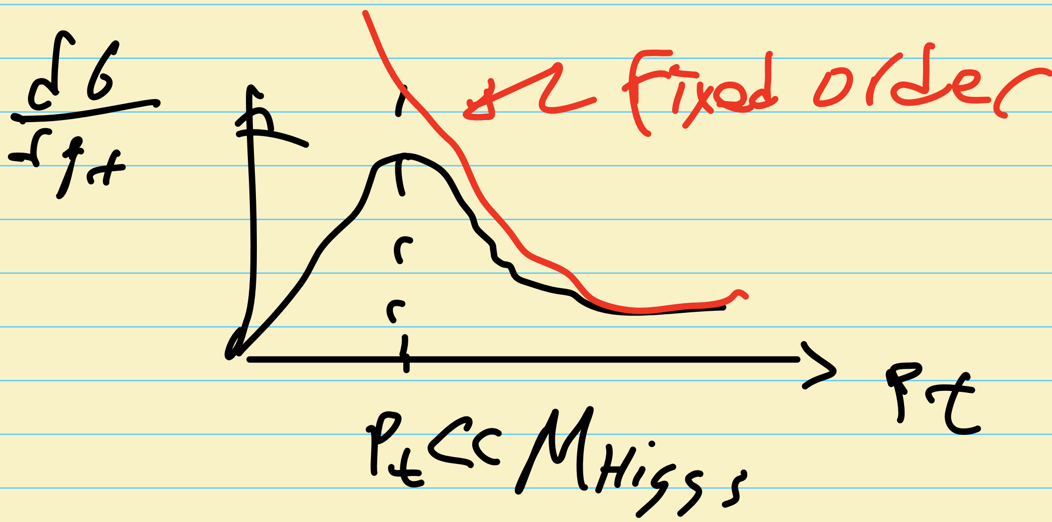
we cannot truncate the α_s series. since $\alpha_s^n L^{2n}$ is equally important.

The logs are usually induced by scale heirachy. Here

hard scale : Q

Jet scale : $Q\delta_s$, $Q\delta_c$

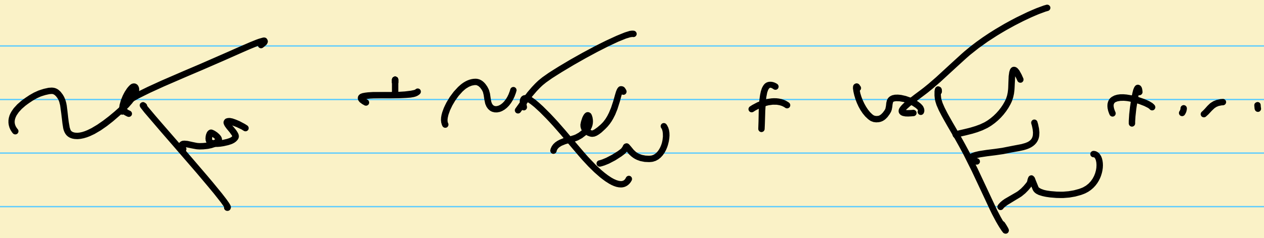
In this case the heirachy is induced by phase-space cut. Other examples including $W/Z/$Higgs p_t distribution



where $\ln \frac{p_t}{M_H}$ will invalidate the fixed-order calculation

the phase space limitation leads to incomplete cancellation of IR singularities, that gives large logs.

One needs to resum these large logs to all orders



which is equivalent to sum up the radiations with same patterns in the soft & collinear limit to all orders. This can be achieved by parton shower or other analytic techniques (RGE)

In other case there are intrinsic hierarchy in the process. for instance in ep, pp collisions, we have

hard : Q
$$\Rightarrow \underline{\underline{\ln \frac{Q}{\Lambda_{QCD}}}}$$
proton : Λ_{QCD}

also as we mention before ~~no~~ way we can be sufficiently inclusive of the initial state in these cases.