Introduction to perturbative QCD

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Part 2. Parton-hadron duality

$\Rightarrow$ Hadron Cross-Section in e^+e^--annihilation

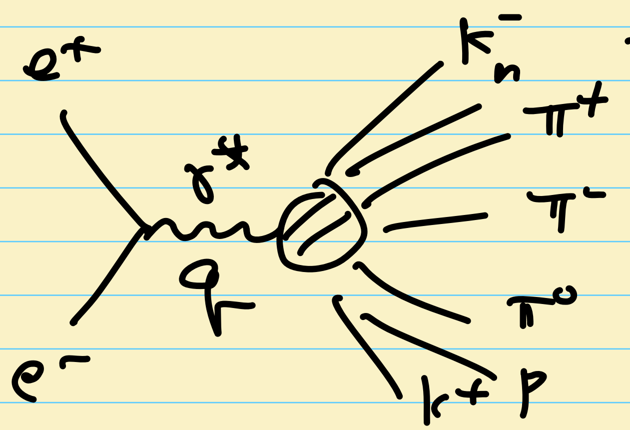
$\Rightarrow$ Operator Product Expansion

$\Rightarrow$ dispersion relation & the parton-hadron duality

∅ Parton-hadron duality

— Hadron X-sec in e^+e^--annihilation

We consider



measurement

incl. (1)

Jet ...

$$\sigma = \frac{e^2}{2s} \int \sum_X \overline{|M|^2}\, \Theta(\Phi_N)$$

phase-space integration

matrix element

phase-space cut.

$$\times (2\pi)^4 \delta^{(4)}\left(q - \sum_X P_X\right)$$

$$= \underbrace{\frac{1}{4}}_{\text{lepton spin ave.}} \frac{e^2}{2s} L_{\mu\nu}(\ell, \bar{\ell}) \rightarrow \text{leptonic tensor}$$

$$\times \underbrace{\frac{1}{(q^2)^2}}_{\text{photon propagator}} \underbrace{(q^2 g^{\mu\nu} - q^\mu q^\nu)}_{\text{required by gauge}} H(q^2)$$

Here

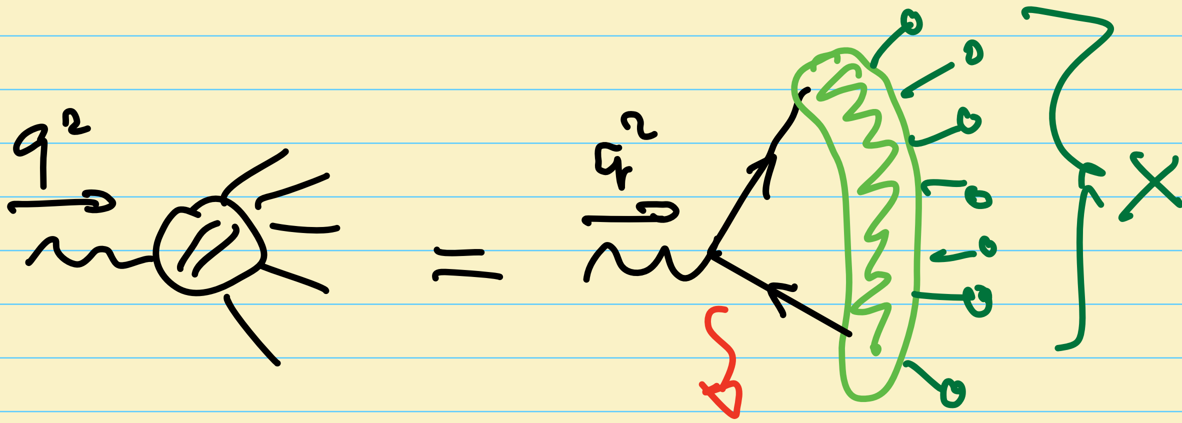
$$L_{\mu\nu}(\ell, \bar{\ell})$$

$$= 4(\ell^\mu \bar{\ell}^\nu + \ell^\nu \bar{\ell}^\mu - g^{\mu\nu} \ell \cdot \bar{\ell})$$

$$\Rightarrow \sigma = \frac{1}{\cancel{4}} \frac{e^2}{2\cancel{s}} \cancel{4} \cancel{q^2} \cdot \frac{(-q^2)}{q^4} H(q^2)$$

$$= \frac{4\pi\alpha}{q^2} \left(-\frac{1}{2}\right) H(q^2)$$

at $q^2 \gg \Lambda_{QCD}^2$ we have



$J^\mu = e\,\bar{q}\gamma^\mu q$

Therefore

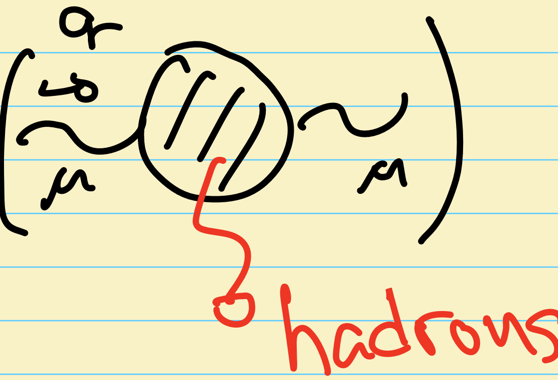
$$3q^2\, H(q^2) = e_q^2 \sum\!\!\!\!\!\!\int_X \langle 0|J^\mu(0)|X\rangle\langle X|J_\mu(0)|0\rangle$$

$$\times (2\pi)^4\, \delta^{(4)}\Big(q - \sum_X P_X\Big)$$

from optical theorem *

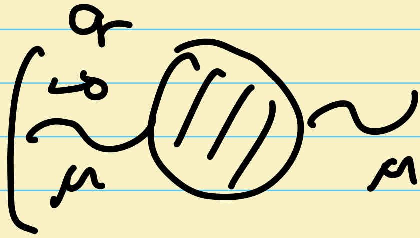
$$3q^2 H(q^2)$$

$$= e_q^2 \times 2\,\mathrm{Im}\, i \int d^4x\, e^{iq\cdot x} \langle 0| T(J^\mu(x) J_\mu(0)) |0\rangle$$

$= e_q^2 \times 2\,\mathrm{Im}\, i \Big($  $\Big)$

and

$$\sigma = \frac{4\pi\alpha}{q^2}\left(-\frac{1}{2}\right)$$

$e_q^2 \times \dfrac{2}{3q^2}\,\mathrm{Im}\, i \Big($  $\Big)$

$$* \quad \langle i|S^\dagger S|i\rangle = 1$$

$$= \langle i|(1 - iT^\dagger)(1 + iT)|i\rangle = 1$$

$$= \cancel{\langle i|i\rangle} - i\langle i|T^\dagger|i\rangle + i\langle i|T|i\rangle + \langle i|T^\dagger T|i\rangle = \cancel{1}$$

$$\Rightarrow \langle i|T^\dagger T|i\rangle = 2\,\mathrm{Im}\langle i|T|i\rangle$$

$$\Rightarrow \sum_f \langle i|T^\dagger|f\rangle\langle f|T|i\rangle = 2\,\mathrm{Im}\langle i|T|i\rangle$$

$$\Rightarrow \sum_f |T_{if}|^2 (2\pi)^4 \delta^{(4)}(i-f) \quad \cancel{(2\pi)^4\delta(0)}$$

$$= 2\,\mathrm{Im}\,T_{ii} \quad \cancel{\delta(0)(2\pi)^4}$$

Specifically, we consider $\checkmark$

$$\int d^4x\, e^{iqx} \langle 0| T[J^\mu(x) J_\mu(0)] |0\rangle$$

$$= \int d^4x\, e^{iqx} \langle 0| J^\mu(x) \; J_\mu(0) |0\rangle\, \theta(x^0)$$

$$+ \int d^4x\, e^{iqx} \langle 0| J^\mu(0) J_\mu(x) |0\rangle \;\theta(-x^0)$$

$$= \int d^4x\, e^{i(q-P_X)\cdot x} \sum_X \langle 0| J^\mu(0) |X\rangle \langle X| J_\mu(0) |0\rangle \int \frac{d\lambda}{2\pi i} \frac{e^{i\lambda x^0}}{\lambda - i\epsilon}$$

$$+ \int d^4x\, e^{i(q+P_X)\cdot x} \sum_X \langle 0| J^\mu(0) |X\rangle \langle X| J_\mu(0) |0\rangle \int \frac{d\lambda}{2\pi i} \frac{e^{i\lambda x^0}}{\lambda + i\epsilon} (-1)$$

$$= \frac{(2\pi)^4}{2\pi i} \frac{\delta^{(3)}(\vec{q} - \vec{P}_X)}{q^0 - P_X^0 - i\epsilon} \sum_X \langle 0| J^\mu |X\rangle \langle X| J_\mu |0\rangle$$

$$- \frac{(2\pi)^4}{2\pi i} \frac{\delta^{(3)}(\vec{q} + \vec{P}_X)}{q^0 + P_X^0 + i\epsilon} \sum_X \langle 0| J^\mu |X\rangle \langle X| J_\mu |0\rangle$$

$\to q^0 + P^0 \neq 0$ for physical states $\Rightarrow i\epsilon \to 0$

$$\Rightarrow 2\,\mathrm{Im}\, i \langle 0| T(J^\mu(x) J_\mu(0)) |0\rangle = \sum_X (2\pi)^4 \delta^{(4)}(q - P_X) |\langle 0| J^\mu |X\rangle|^2$$

- Operator product Expansion (OPE)

now if $x \to 0$, more precisely $x \ll \frac{1}{\Lambda_{QCD}}$. we can expand $J(x)J(0)$ in terms of local operators

$$\overbrace{\frac{1}{3q^2}}^{\text{dim-}(-2)}\overbrace{\int d^4x}^{\text{dim-}(-4)}\, e^{iq\cdot x}\cdots\overbrace{J^\mu(x)J_\mu(0)}^{\text{dim-}6}\cdots \qquad \text{dim-}0$$

$$= \frac{1}{3q^2}\int d^4x\, e^{iq\cdot x}\cdots\Big\{C_{\mathbb{1}}(x)\,\mathbb{1} \leftarrow \text{dim-}0$$

$$+ C_q(x)\,\underline{m\bar{q}q(0)} + C_g(x)\,F^2 \leftarrow \text{dim-}4$$

$$+ \text{dim-6 operators} + \cdots\Big\}\cdots$$

⇒ as long as symmetries allow,
the operator will occur on the
right in the expansion.
some of them may not be independent
e.g. related by equations of motion
Independent operators form a basis

⇒ $\langle 0|\cdots|0\rangle$ ⇒ only scalars.

⇒ organized by the dimension
of the operators

$$= \tilde{C}_1(q^2)\,\langle 0|\,\mathbb{1}\,|0\rangle$$

↳ parton

$$+\,\tilde{C}_q(q^2)\,\frac{1}{q^4}\,\langle 0|m\bar{q}q|0\rangle$$

$=$ cf $\mathcal{O}(\Lambda^4_{QCD})$

$$+\,\tilde{C}_g(q^2)\,\frac{1}{q^4}\,\langle 0|F^2|0\rangle$$

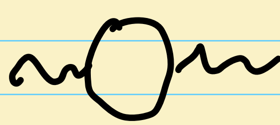
dim - 0

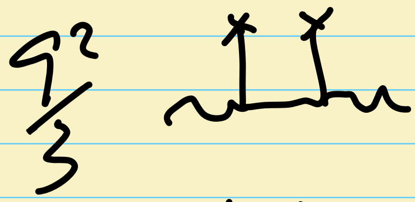
↳ $\mathcal{O}(\Lambda^4_{QCD})$

$\Rightarrow$ one type of factorization

$\Rightarrow$ Short-distance physics in the coefficient $\tilde{C}_i(q)$, Wilson Coefficient

can be perturbatively calculated !

e.g.  $\sim \tilde{C}_1(q^2) = \Pi(q^2)$

 $\sim \quad \tilde{C}_q(q^2)$

 $\sim \quad \tilde{C}_g(q^2)$

→ long-distance in the $\langle 0|\cdots|0\rangle$, knows about hadrons, but suppressed by $\frac{\Lambda^4_{QCD}}{q^4}$

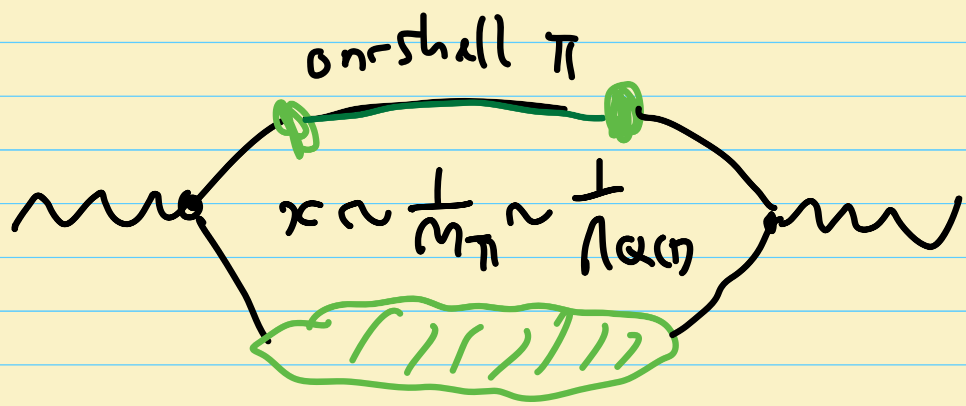
→ higher dimensional operators are less important/more suppressed as $q^2 \to \infty$

If the OPE is allowed, then the hadron cross-section is determined dominantly by $C_1(q^2)$ which knows only quarks and gluons and can be perturbatively calculated

But the OPE is valid when $q^2 < 0$ (space-like)

in this region, no on-shell hadron can be produced, $x \sim \frac{1}{q} \ll \frac{1}{\Lambda_{QCD}}$

$\llcorner q^2 > 0$

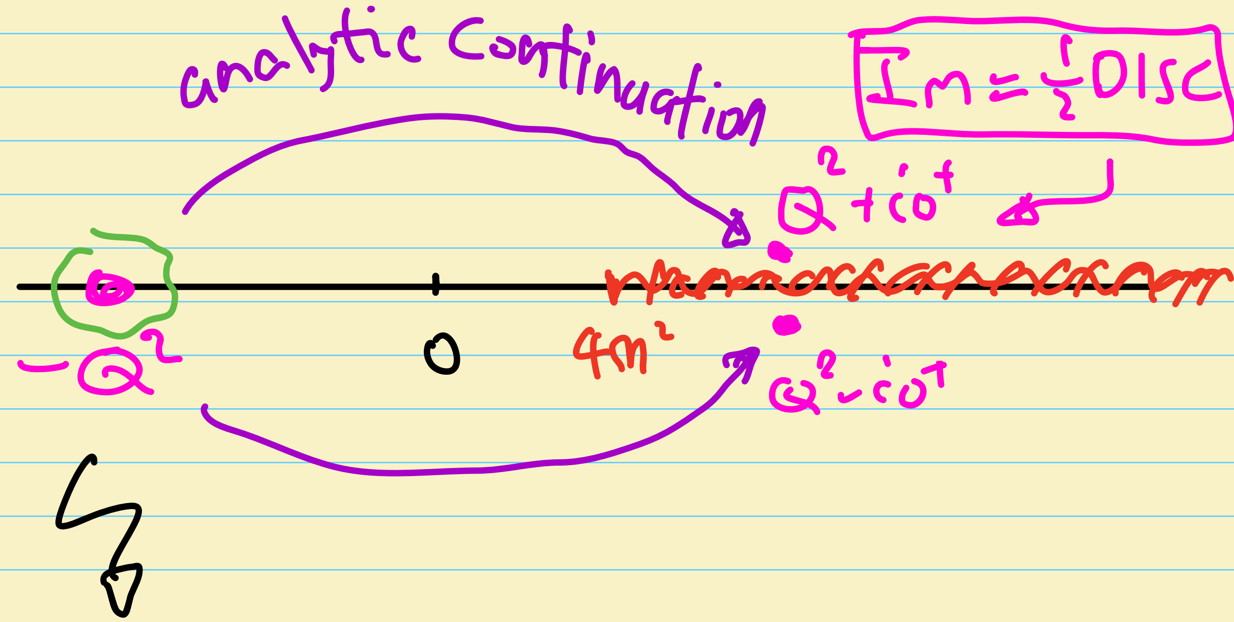


OPE is NOT justified in the physical region.

Typically what we do is performing OPE for $q^2 < 0$. and analytically continuing to the physical region $q^2 > 0$

$\llcorner q^2$

analytic everywhere, except $\mathrm{Re}(q^2) > 4m^2$



Perform OPE $H(-Q^2) = e_q^2\, \Pi(-Q^2) + \frac{\Lambda^4_{QCD}}{Q^4}$

However since we do not know the complete form of $\Pi(-Q^2)$ we can only do the analytic continuation order-by-order. We miss information

For instance

$$\Pi(q^2) = \frac{\alpha}{3\pi}\left\{\ln\frac{-q^2+i0^+}{\mu^2} - 5/3\right\} N_c$$

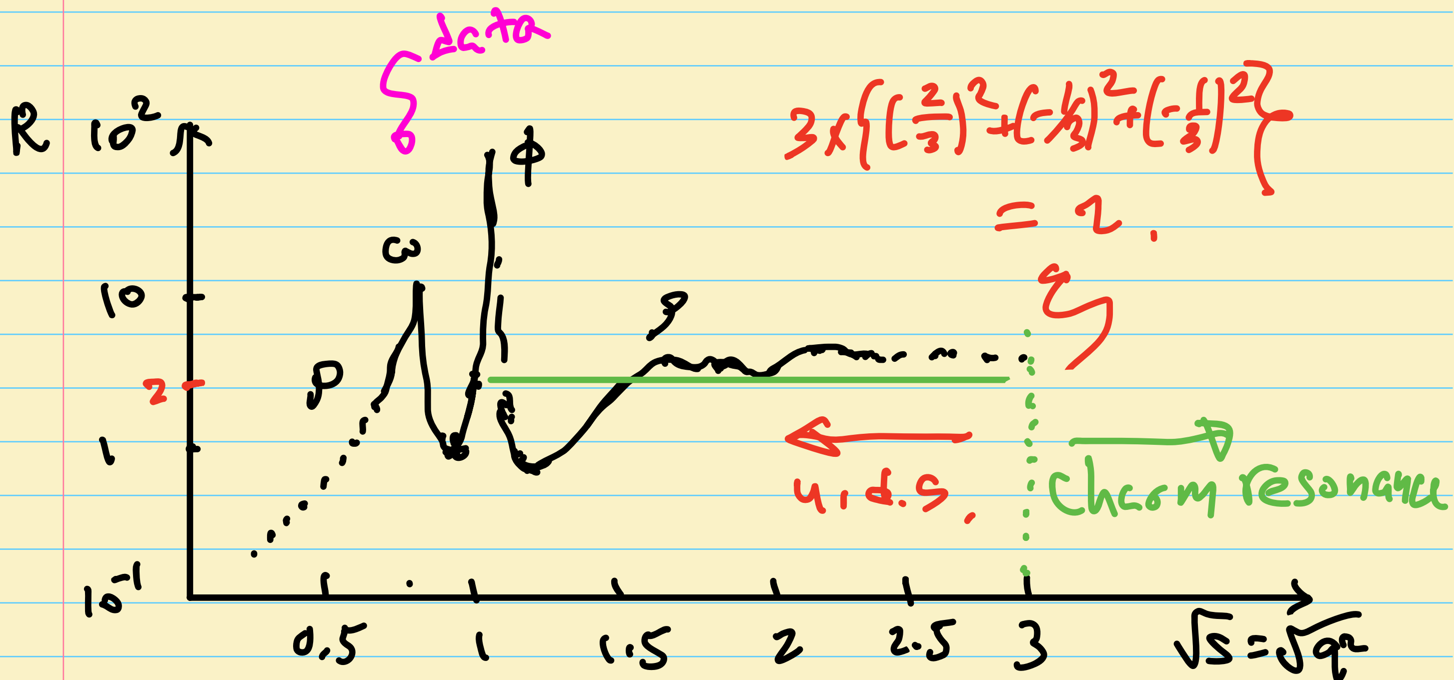
$$\mathrm{Im}(-\Pi(q^2)) = -\frac{\alpha}{3\pi}\pi N_c = -\frac{\alpha}{3} N_c$$

$$\sigma = \frac{4\pi\alpha}{q^2}\left(-1/2\right) \quad e_q^2 \times 2\left(-\frac{\alpha}{3}N_c\right)$$

$$= \sum_q \frac{4\pi\alpha^2 e_q^2 N_c}{3s} + \mathcal{O}(\alpha_s)$$

Exceperimentally, one usually looks at

$$R = \frac{\sigma_{hadron}}{\sigma_{\mu^+\mu^-}} \simeq \sum_q e_q^2 N_c + \mathcal{O}(\alpha_s) + \cdots$$



good agreement between data & pQCD at large q^2.

actually LO pQCD is lower than the measured R value. which can be improved by higher order corrections as we will see in part 3 of this lecture

In the resonance region, the data exhibits peaks & oscillation. $\neq$ pQCD calculation

– dispersion relation &
the parton hadron duality.

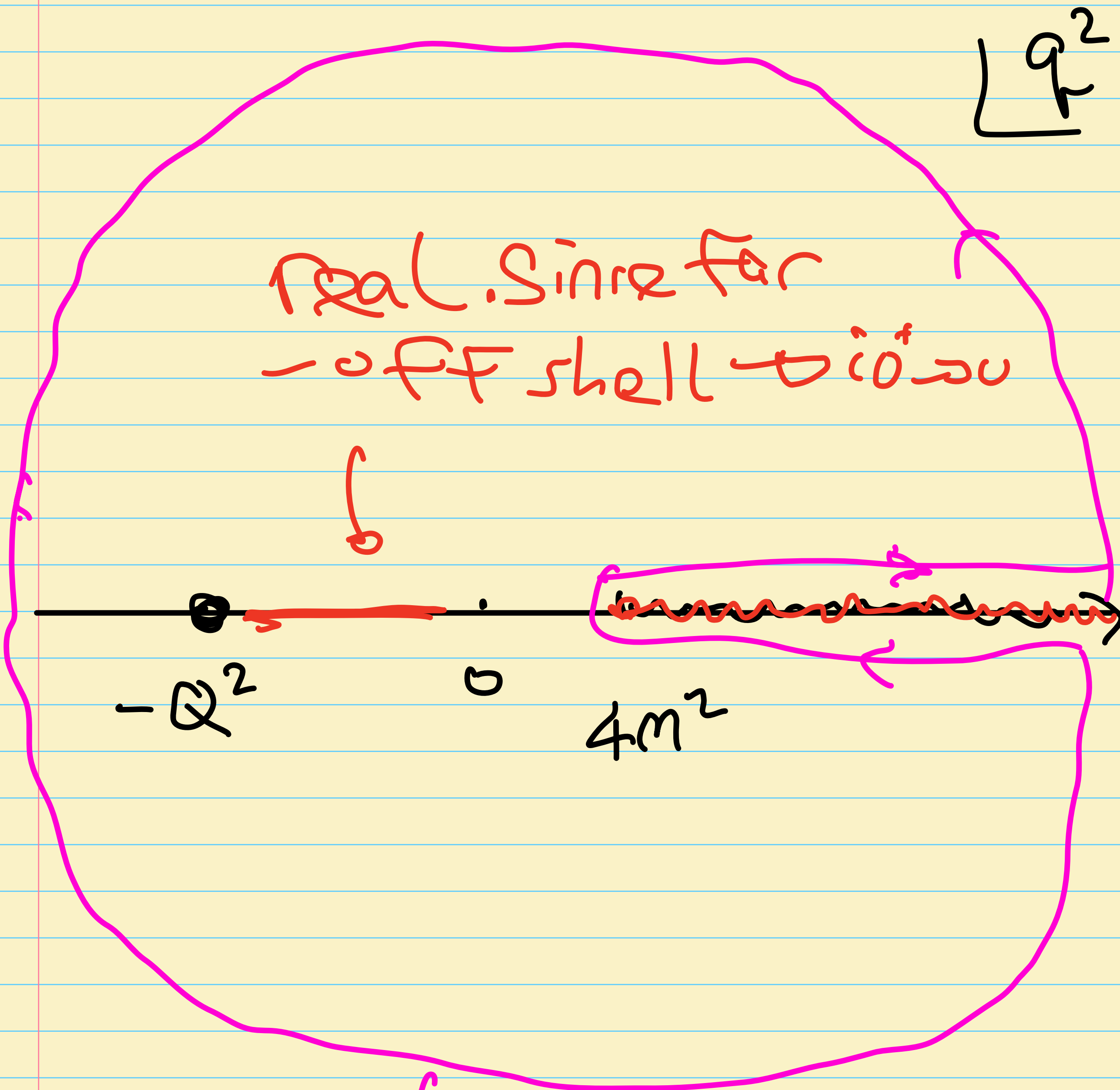
To have a better understanding of the previous comparison between data of pQCD prediction, we consider the contour integration

$$\frac{1}{2\pi i} \oint dq^2 \frac{A(q^2)}{(q^2+Q^2)^2} = \frac{d}{dq^2} A(q^2) \Big|_{q^2=-Q^2}$$

where

$$A(q^2) \equiv i \int d^4x\, e^{iq\cdot x} \langle 0 | T\{ J^\mu(x) J_\mu(0) \} | 0 \rangle e_q^2$$

and the contour is given by



$\dfrac{A(q^2)}{(q^2+Q^2)^2}$ decays fast enough to vanish on the boundary.

The contour indicates that

$$\left.\frac{d}{dq^2}A(q^2)\right|_{q^2=-Q^2}$$

OPE valid

only quarks & gluons.

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{A(q^2+i0^+)-A(q^2-i0^+)}{(q^2+Q^2)^2}\,dq^2$$

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{A(q^2)-A^*(q^2)}{(q^2+Q^2)^2}\,dq^2$$

Schwartz reflection principle

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{2i\,\mathrm{Im}\,A(q^2)}{(q^2+Q^2)^2}$$

optical Theorem

$$= \underbrace{\frac{1}{2\pi}\int_{4m}^{\infty}\frac{q^2\sigma_{e^+e^-\to X}(q^2)}{(q^2+Q^2)^2}\,dq^2 \times \frac{-2}{4\pi\alpha}}_{\text{weighted hadronic crosssection}}$$

The dispersion relates the weighted hadronic cross-section with the OPE formalism in terms of partons.

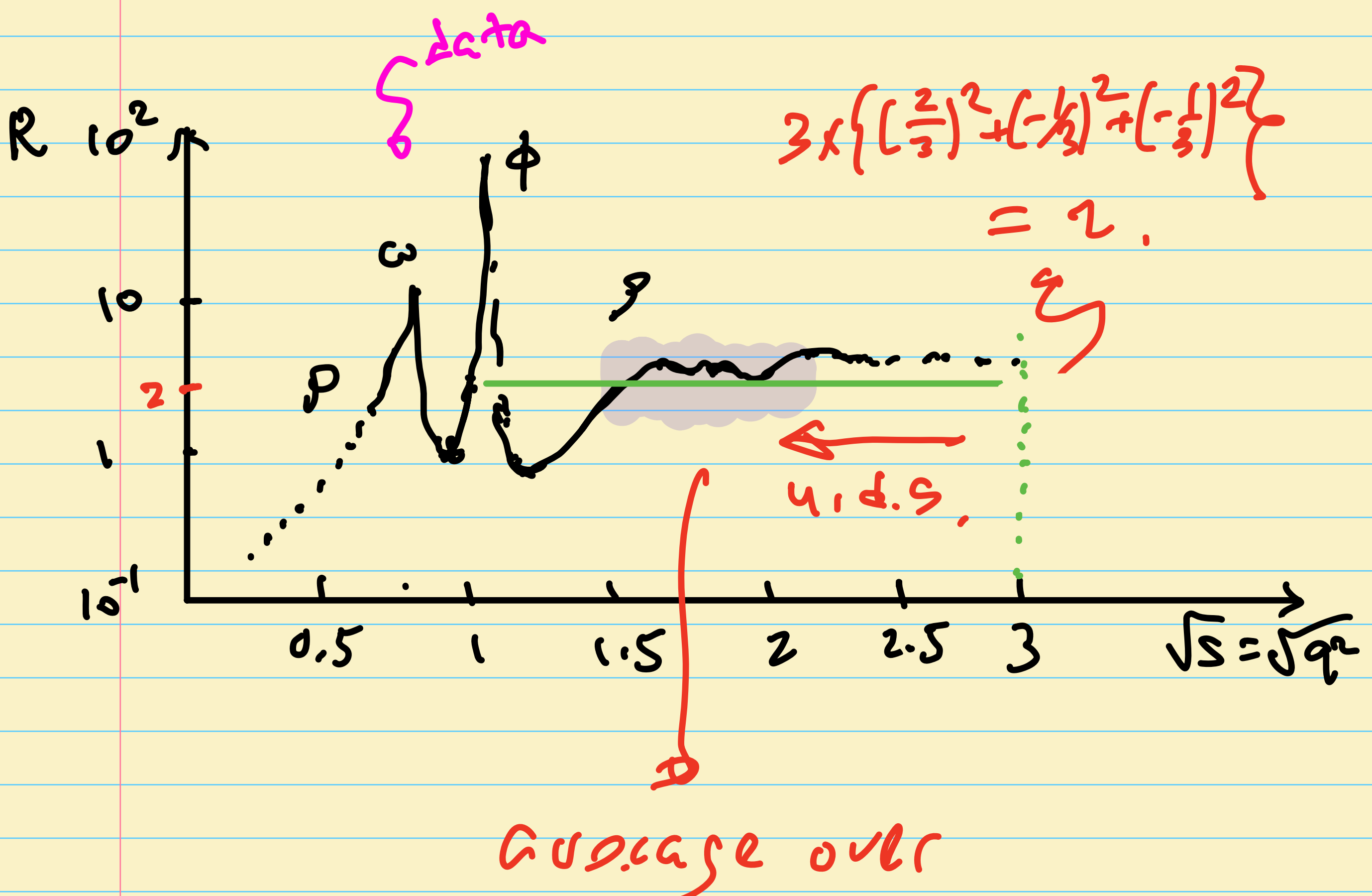
The morality of the relation is that:

If we are sufficiently inclusive over the hadrons, then the calculation using partons can approximate well the hadronic observables.

$\Longrightarrow$ Parton-hadron duality.

$\Rightarrow$ in low q^2 region or at the resonance it produces an exclusive hadron

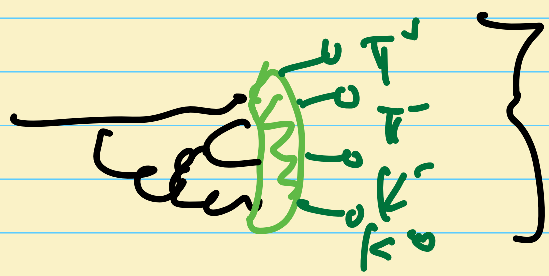
the partonic calculation fails to describe the data. In order for pQCD to work, we need to average over the q^2 region to cover sufficiently inclusive over the hadrons



$\Rightarrow$ at large values of $q^2 \gg \Lambda_{QCD}^2$

the cross-section already includes sufficient hadrons. therefore the parton-hadron duality holds locally in q^2

an intuitive way to understand this is via probability



all measured

if inclusive enough

Prob(producing partons) $\simeq$ Prob(producing hadrons)

⇒ examples of "sufficiently Inclusive"

- Inclusive cross-section (Z/H p_t dist.)
- jets (bag of hadrons)
- event shapes (thrust, . . .)

can be understood perturbatively using partons

⇒ examples of "insufficiently inclusive"

- tag a hadron in the final state, but inclusive over others $h+X$ (semi-inclusive)
- initial state hadron, e.g. ep. pp

additional non-pert. distribution functions are required. PDFs & fragmentation functions.

crucial for the LHC