Introduction to perturbative QCD

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Why (perturbative) QCD ?

- QFT, $\cancel{CP}$ & Flavor.

  nuclear physics, $\cdots$

- $\alpha_s(M_Z) \gtrsim 0.1 \gg \alpha \sim \alpha_W \sim 10^{-2}$

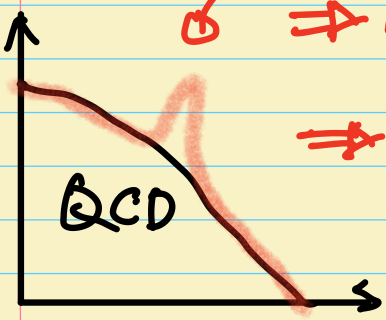
  $\Rightarrow$ dominant activities at colliders

  $\Rightarrow$ dominant backgrounds for new physics searches.

See a bump

$\Rightarrow$ new physics

$\Rightarrow$ 

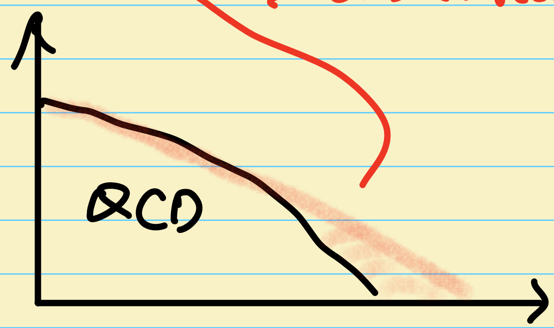


ideal

excess

$\Rightarrow$ N.P.?

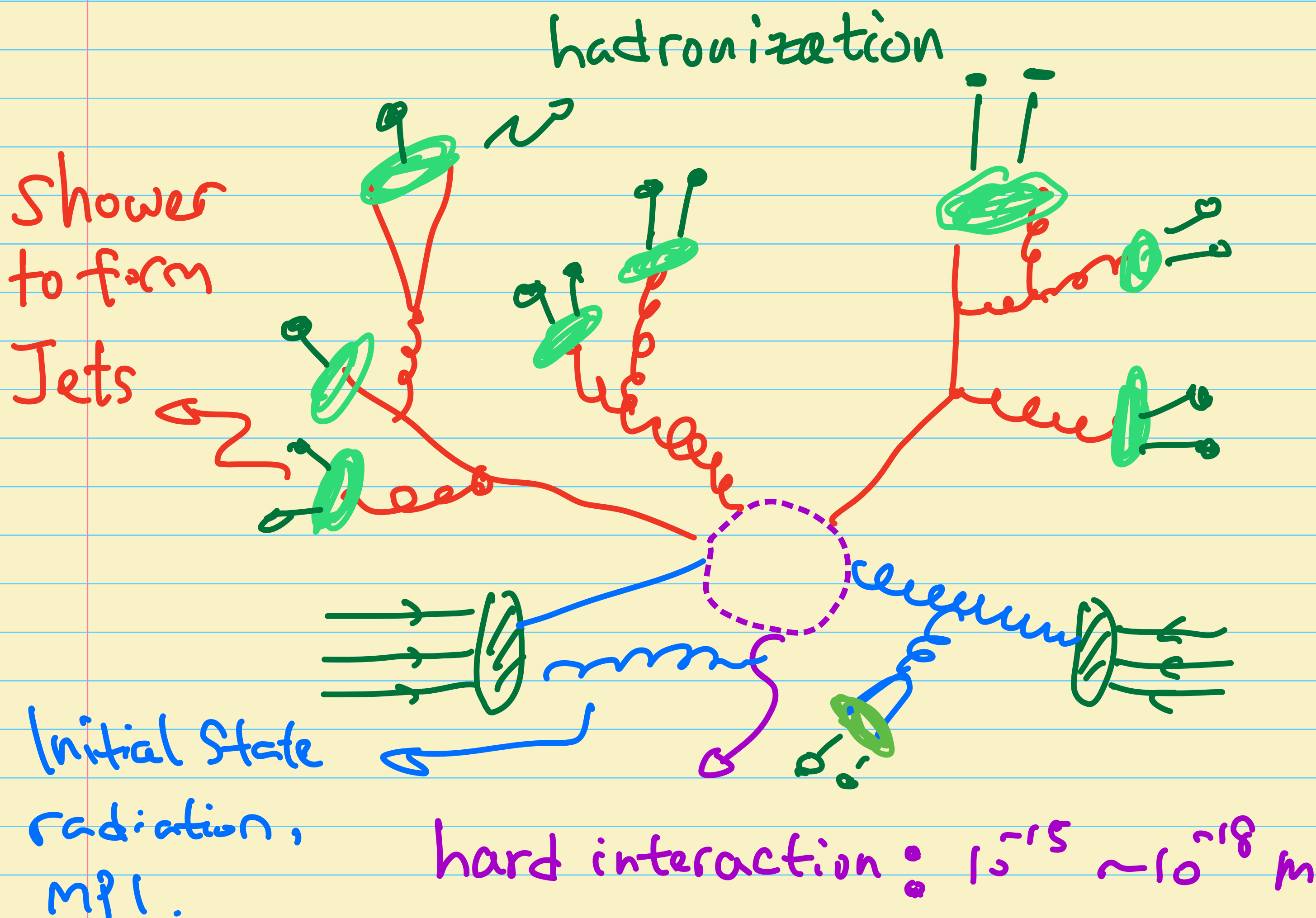
$\Rightarrow$ QCD correction?



reality

new physics = Data − QCD

a typical event at the LHC



Factorization

$\sigma \sim f_{i/p}\, f_{j/p}\, \hat{\sigma}_{ij \to H/W/Z \cdots}$

$\hat{\sigma}$: partonic cross-section
involving quarks/gluons
can be calculated perturbatively

$f_{i/p}$: parton distribution function
(PDF), non-perturbative

Prob. to pick a parton i
from the proton.

Part 1. Basics of QCD

$\Rightarrow$ QCD Lagrangian

$\Rightarrow$ Running Coupling &
asymptotic freedom

∅ Basics of QCD

- SU(3) gauge theory

3 ~ # of colors

$$\mathcal{L} = -\frac{1}{4} G^{A}_{\mu\nu} G^{\mu\nu A}$$ ~ gauge fields: gluon

$$+ \sum_{i=udscbt} \bar{q}^{a}_{i} (i\not{D} - m_i)_{ab}\, q^{b}_{i}$$ ~ quarks

+ gauge fixing + topologic term ~ ignore

+ Counter terms ~ renormalization

Here

strong coupling

$$D_{ab} = \partial_\mu \delta_{ab} - i g_s t^A_{ab} A^A_\mu$$

color index : 1, 2, 3

Here. t^A satisfies

$$\underbrace{[t^A, t^B]_{ab}}_{\text{fundamental rep. of } SU(3)} = i \underbrace{f^{ABC}}_{\text{adjoint representation}} t^C_{ab}$$

normalization

$$\mathrm{tr}[t^A t^B] \equiv \frac{1}{2} \delta_{AB}$$

Fierz identity

$$t^A_{ij} t^A_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ik}\delta_{jl}\right)$$

gluon field, $1 \to 8$

$$G^A_{\mu\nu} \equiv \underbrace{\partial_\mu A^A_\nu - \partial_\nu A^A_\mu}_{\text{similar to QED}} - \underbrace{g_s f^{ABC} A^B_\mu A^C_\nu}_{\text{non-abelian part}}$$

note that except for top,

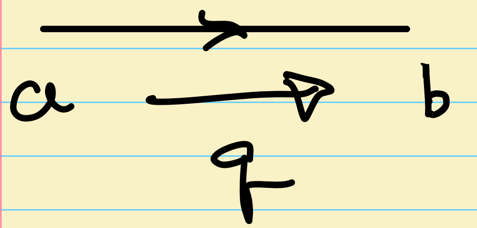
$m_i \lesssim 3\,\text{GeV}$ for u, d, s, c, b

which is small compare with

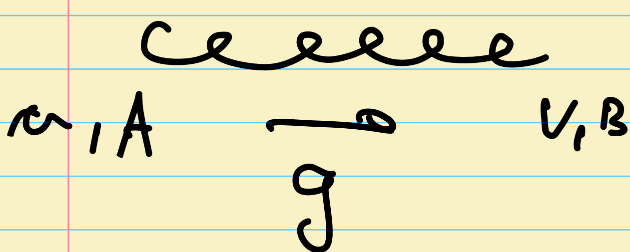
$\sqrt{s} \sim \mathcal{O}(100)$ GeV. Therefore

$\boxed{m_i \sim 0}$ at high energy colliders

– Feynman Rules:



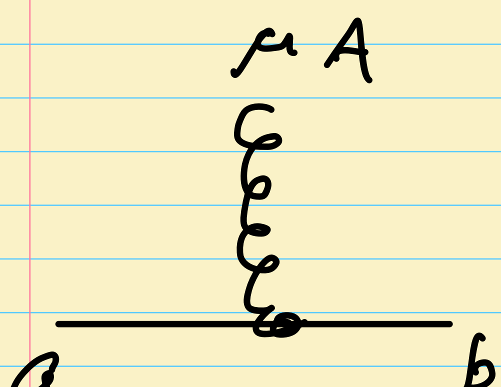
$$= \frac{i(\not{q} + m_i)}{q^2 + i0^+}\,\delta_{ab}$$



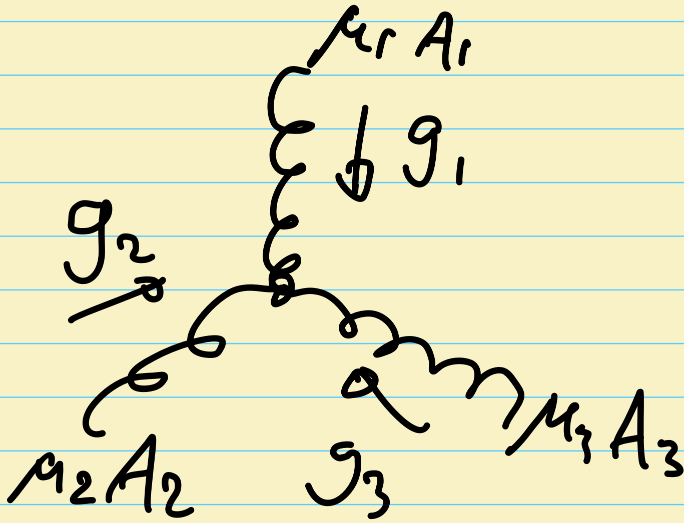
$$= \frac{-i\, g^{\mu\nu}}{q^2 + i0^+}\,\delta_{AB}$$

Feynman gauge

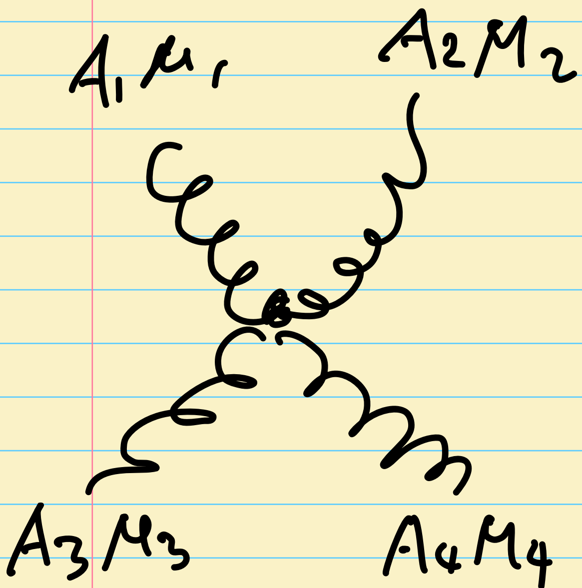
axial gauge $= \frac{-i}{q^2}\left[g^{\mu\nu} - \frac{n^\mu q^\nu + n^\nu q^\mu}{n\cdot q} + \frac{n^2 q^\mu q^\nu}{(n\cdot q)^2}\right]$



$$= i g_s t^A_{ab}$$



$$
= g_s f^{A_1 A_2 A_3} \Big[g^{\mu_1 \mu_2} (g_1 - g_2)^{\mu_3} + 1 \to 2 \to 3 \to 1 + 1 \to 3 \to 2 \to 1 \Big]
$$

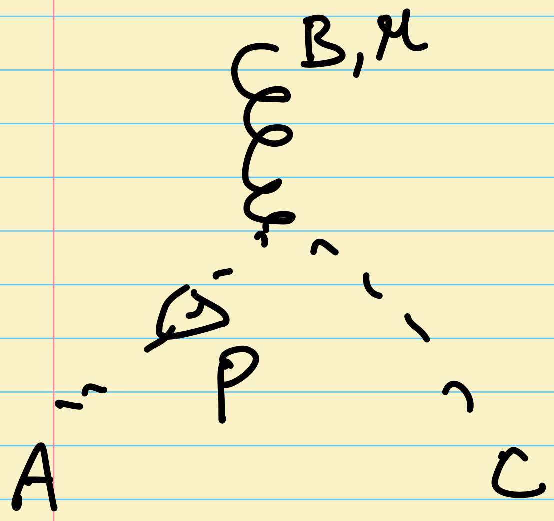


$$= -ig_s^2 \left\{ f^{A_1 A_2 B} f^{A_3 A_4 B} \left(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \right) \right.$$

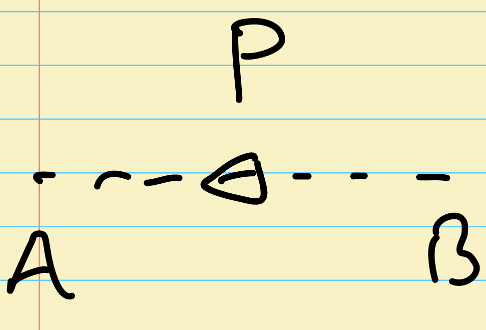
$$+ \quad 2 \leftrightarrow 3$$

$$\left. \begin{array}{l} 2 \leftrightarrow 3 \\ + \text{ then } 3 \leftrightarrow 4 \end{array} \right]$$

ghost.



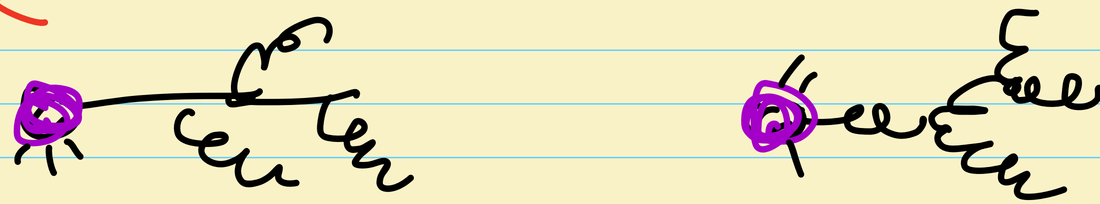
$$= -g_s f^{ABC} p^\mu$$



$$= \frac{i\delta_{AB}}{p^2 + i0^+}$$

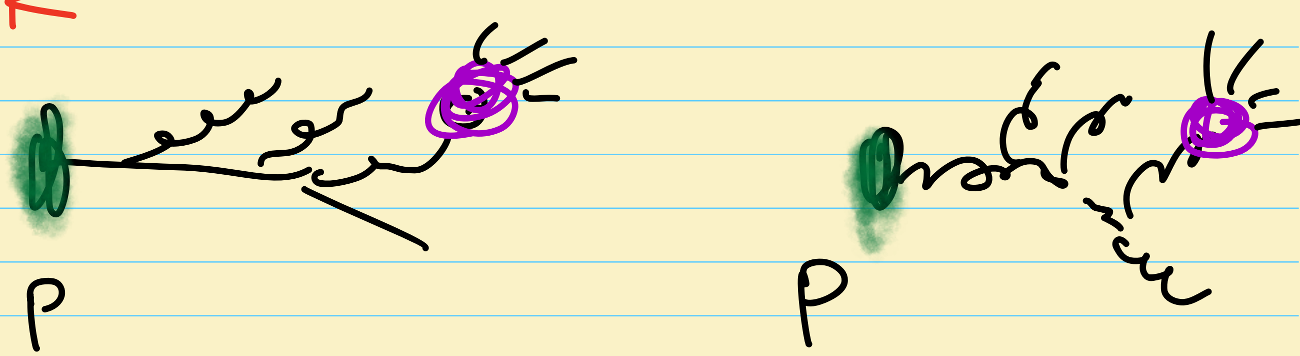
as already seen from the QCD Feyn. rules, both quarks & gluons couple to gluons

FSR



"everything" eventually decays to gluons

ISR

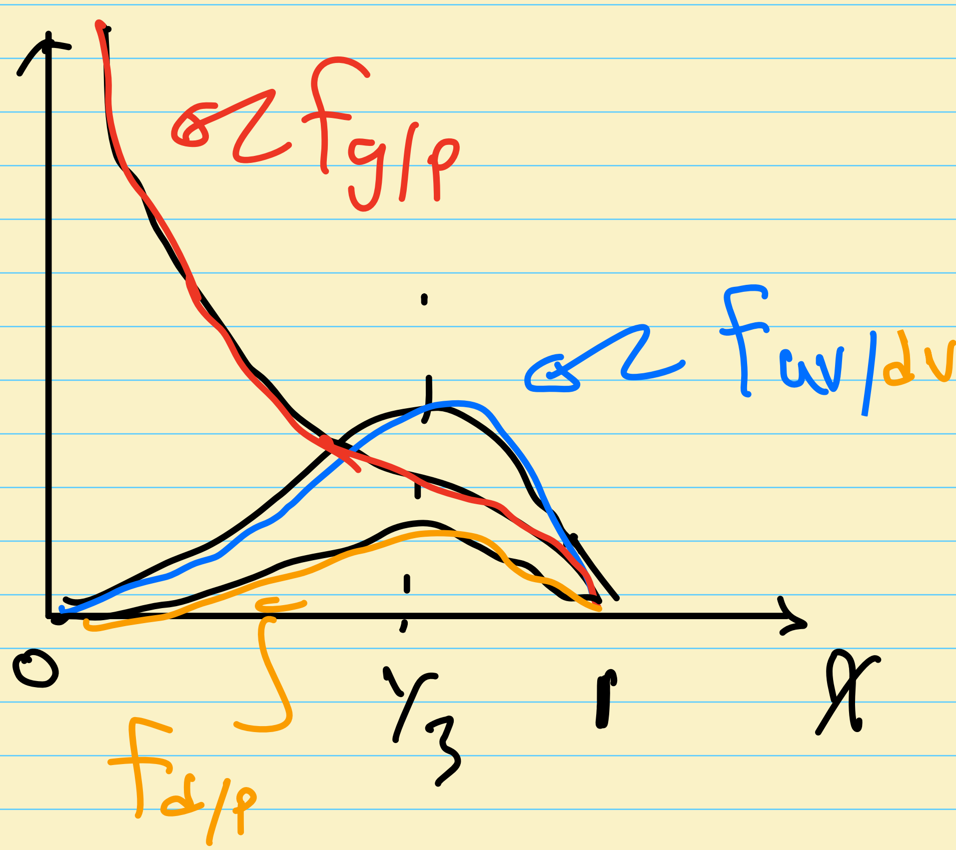


more chance to pick a g in a proton

typically more gluons at
a high energy collider.

e.g.

$x f_{i/p}(x)$ at large machine energy

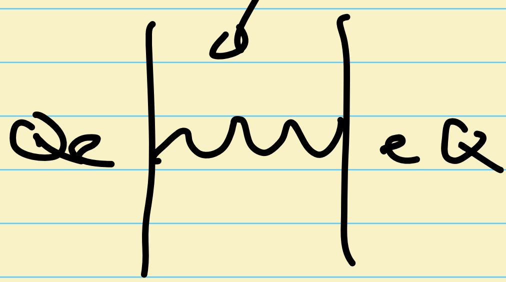


$\Rightarrow$ gluon channel will dominate

- Color charge:

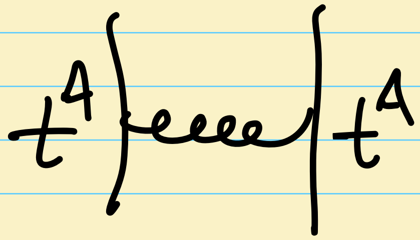
QED.

exchanging photon



$$\sim \underbrace{Q^2}_{\text{electromagnetic charge}} e^2 \sim \alpha Q^2$$

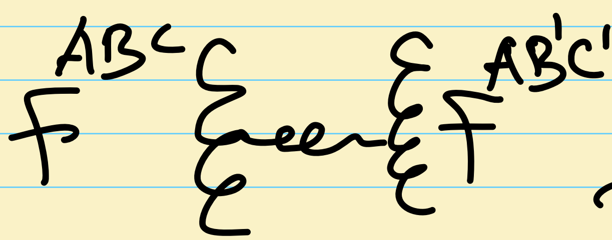
QCD



$$\sim g_s^2\, t^A t^A$$

$$= g_s^2\, C_F\,, \qquad C_F = 4/3$$

Color charge for quarks



$$\sim g_s^2\, f^{ABC} f^{AB'C'}$$

$$\sim g_s^2\, C_A\,, \quad C_A = 3$$

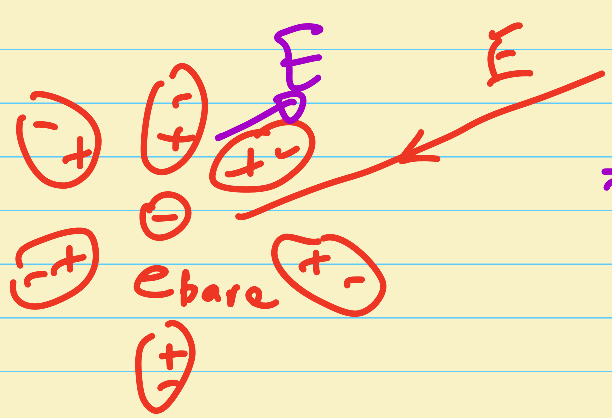
Color charge for gluons

– Running coupling
& Asymptotic Freedom

the strength of the strong coupling $\alpha_s \equiv \frac{g_s^2}{4\pi}$. depends on the length scale we probe

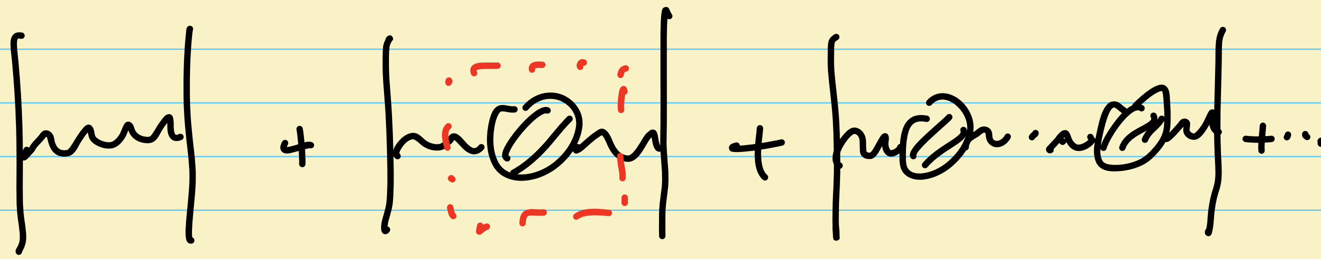
QED

→ screening of the E&M field

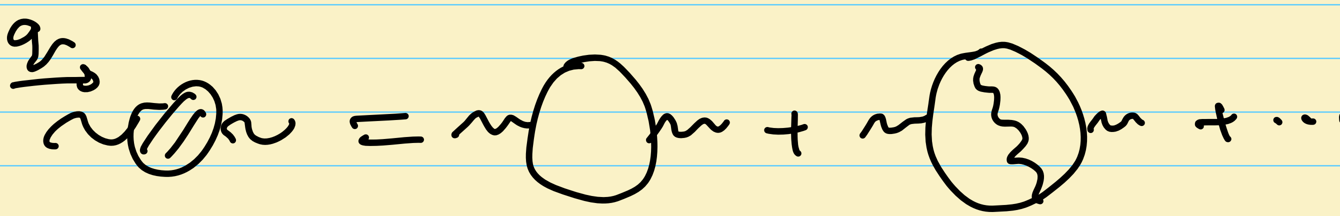


⇒ effectively α reduced at large distance

This can be made rigorously by calculating

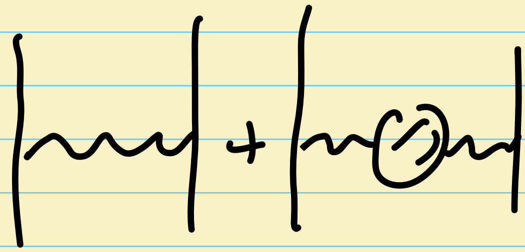


Vacuum polarization



$$= i\underbrace{\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)}_{\text{required by gauge symmetry}} \Pi(q^2)$$

q_μ  $= 0$

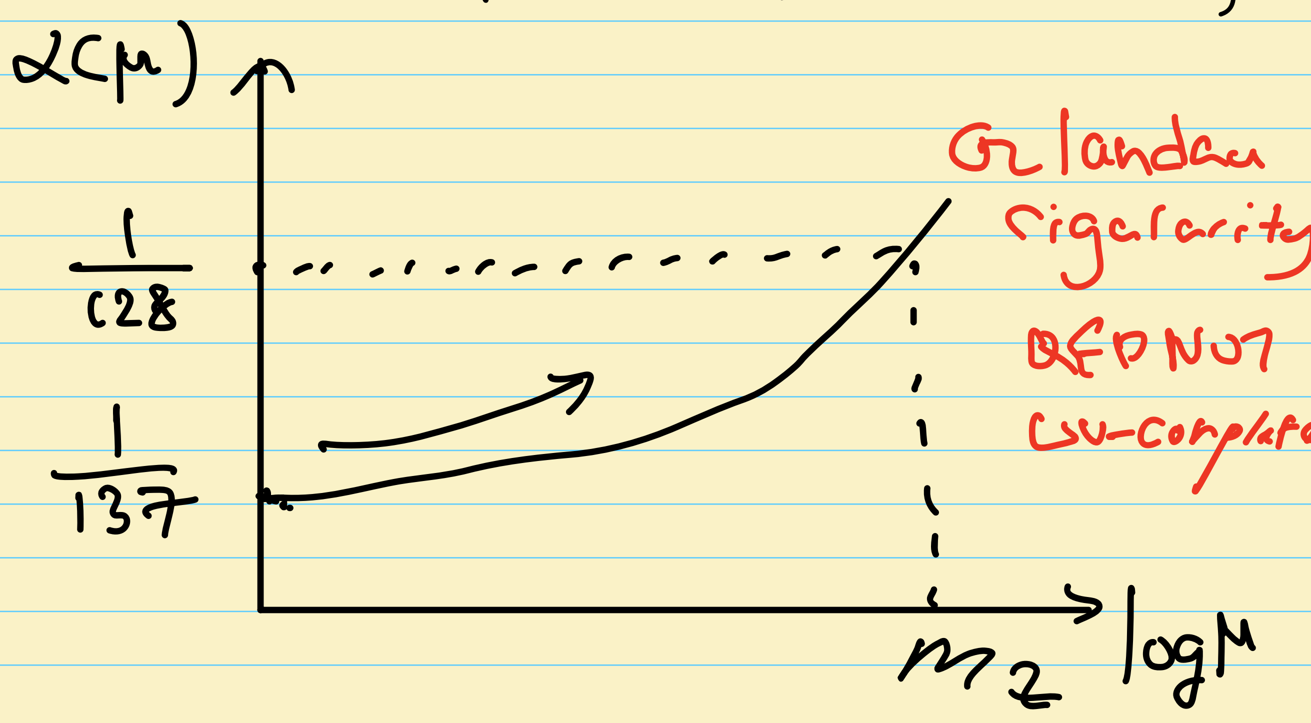
$$\Rightarrow$$  $$+ \cdots = \frac{e^2(\mu)}{1-\Pi(q^2)} \; \frac{-i g^{\mu\nu}}{q^2}$$

effective coupling $\alpha(q^2)$

Columb potential

$$\frac{d\alpha(\mu)}{d\ln\mu} = \beta[\alpha] \overset{\text{1-loop}}{\simeq} \frac{2\alpha^2}{3\pi} > 0 \quad *$$

$$\Pi^{(2)}_{\text{ren}}(q^2) = \frac{\alpha}{3\pi} \cdot \left\{ \ln\frac{-q^2}{\mu^2} - 5/3 \right\}$$



* Derivation

renormalization in QED

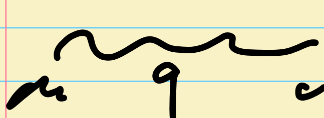
$$A_{\mu,0} = Z_A^{1/2} A_\mu , \quad e_0 = Z_\alpha^{1/2} e$$

where "0" stands for the bare quantity

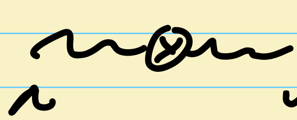
In dim-reg, $\overline{MS}$ scheme, we have

$$\frac{e_0^2}{(4\pi)^{d/2}} = \mu^{2\epsilon}\frac{\alpha(\mu)}{4\pi} Z_\alpha(\alpha(\mu))\, e^{\gamma_E \epsilon} \quad \cdots (1)$$

The feyn Rule for the photon propagator is then



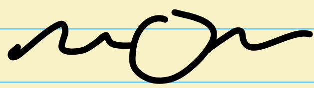
$$= -i D_0^{\mu\nu} = -i\frac{g^{\mu\nu}}{q^2+i\epsilon}$$



$$= -\delta Z_A \left(-i D_0^{\mu\nu}\right)$$

where $Z_A = 1 + \delta Z_A$

at one-loop. $d = 4-2\epsilon$



$$= -i D_0^{\mu\nu} \times (-8) \frac{e_0^2}{(4\pi)^{d/2}} (-q^2)^{-\epsilon}$$

$$\times \frac{\Gamma(2-d/2)}{\Gamma(d)} \Gamma^2(d/2)$$

$$= -i D_0^{\mu\nu} \times (-8) \frac{\alpha(\mu)}{4\pi} Z_\alpha \left(\frac{-q^2}{\mu^2}\right)^{\epsilon} e^{\gamma_E \epsilon}$$

$$\times \frac{\Gamma(2-d/2)}{\Gamma(d)} \Gamma^2(d/2)$$

$$= -i D_0^{\mu\nu} \left\{ -\frac{\alpha(\mu)}{3\pi\epsilon} + \frac{\alpha(\mu)}{3\pi}\left[\log\frac{-q^2}{\mu^2} - 5/3\right]\right\}$$

$\Rightarrow$ in the $\overline{MS}$ scheme

$$Z_A = 1 + \delta Z_A = 1 - \frac{\alpha(\mu)}{3\pi\epsilon}$$

QED Ward-identity ensures that

$$Z_\alpha^{-1} = Z_A$$

Therefore

$$Z_\alpha = 1 + \frac{\alpha(\mu)}{3\pi\epsilon}$$

Now we take derivative $d/d\ln\mu$ to Eq.(1) to find

$$0 = 2\epsilon\,\mu^{2\epsilon}\,\frac{\alpha(\mu)}{4\pi}\,Z_\alpha\,e^{\gamma_E\epsilon}$$
$$+\ \mu^{2\epsilon}\,\frac{1}{4\pi}\,\frac{d\alpha(\mu)}{d\ln\mu}\,Z_\alpha\,e^{\gamma_E\epsilon}$$
$$+\ \mu^{2\epsilon}\,\frac{1}{4\pi}\,\frac{\alpha(\mu)}{4\pi}\,\frac{dZ_\alpha}{d\ln\mu}\,e^{\gamma_E\epsilon}$$

which gives

$$\frac{d\alpha(\mu)}{d\ln\mu} = \left(- \frac{d\ln Z_\alpha}{d\ln\mu} - 2\epsilon \right)\alpha(\mu)$$

at one-loop

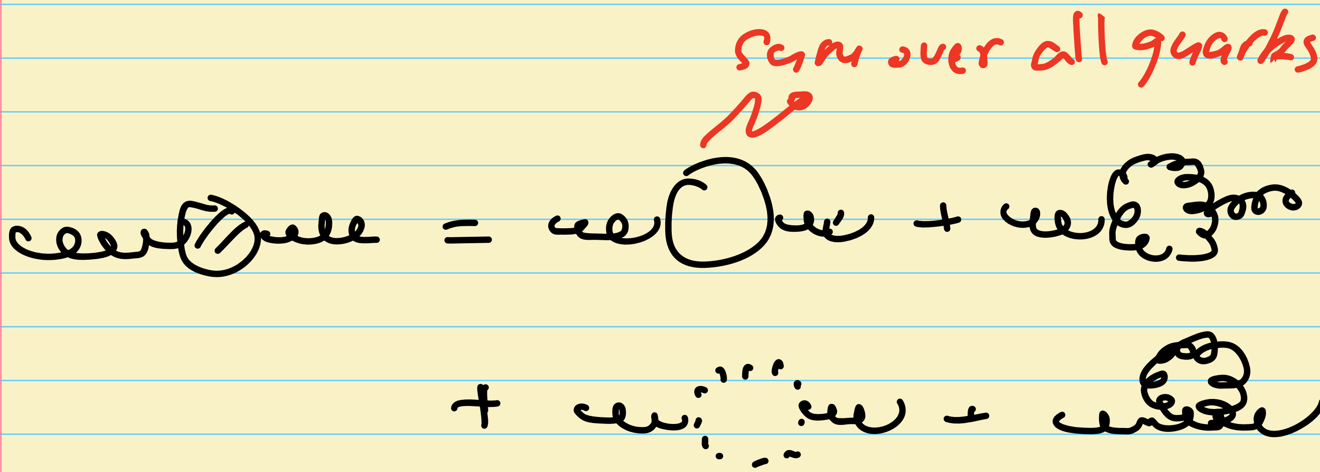
$$\frac{d\ln Z_\alpha}{d\ln\mu} = \frac{1}{3\pi\epsilon}\,\frac{d\alpha(\mu)}{d\ln\mu} \sim O(\alpha(\mu))$$

$$\Longrightarrow \frac{d\alpha(\mu)}{d\ln\mu} = \left(- \frac{1}{3\pi\epsilon}\,\frac{d\alpha(\mu)}{d\ln\mu} - 2\epsilon \right)\alpha(\mu)$$

$$= \left(- \frac{1}{3\pi\epsilon}\left\{ - \underbrace{\cancelto{0}{\frac{d\ln Z_\alpha}{d\ln\mu}}}_{\sim O(\alpha)} - 2\epsilon \right\}\alpha(\mu) - \cancelto{0}{2\epsilon} \right)\alpha(\mu)$$

$$\xrightarrow{\text{1-loop}} \quad \frac{d\alpha(\mu)}{d\ln\mu} = \frac{2}{3\pi}\alpha^2(\mu) \quad > 0$$

Similar Story happens in QCD.
but the gluon self interaction provides the anti-screening.



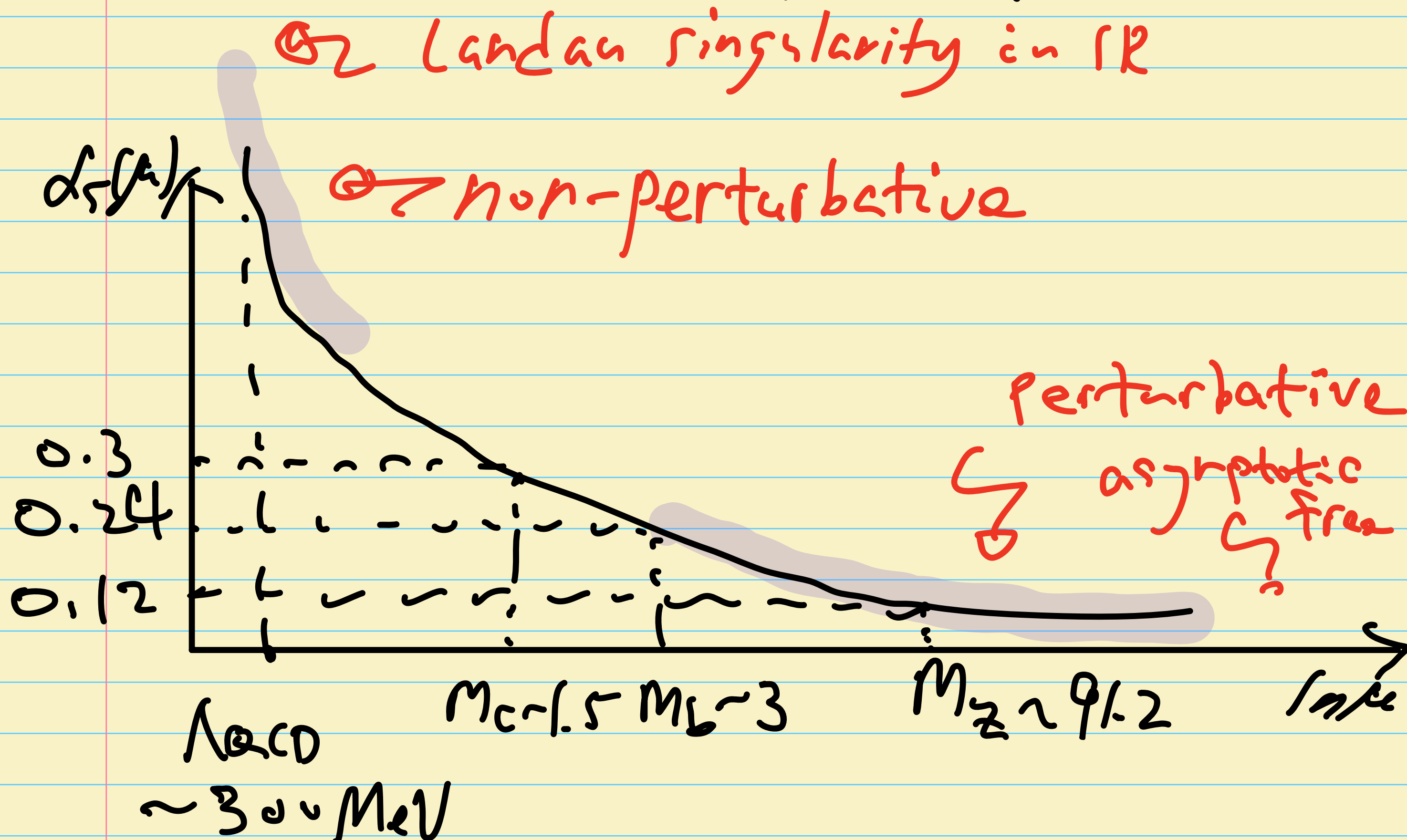
$$\Rightarrow \frac{d\alpha_s(\mu)}{d\ln\mu} = \beta[\alpha_s] = -2\alpha_s \sum \beta_i \left(\frac{\alpha_s}{4\pi}\right)^{i+1}$$

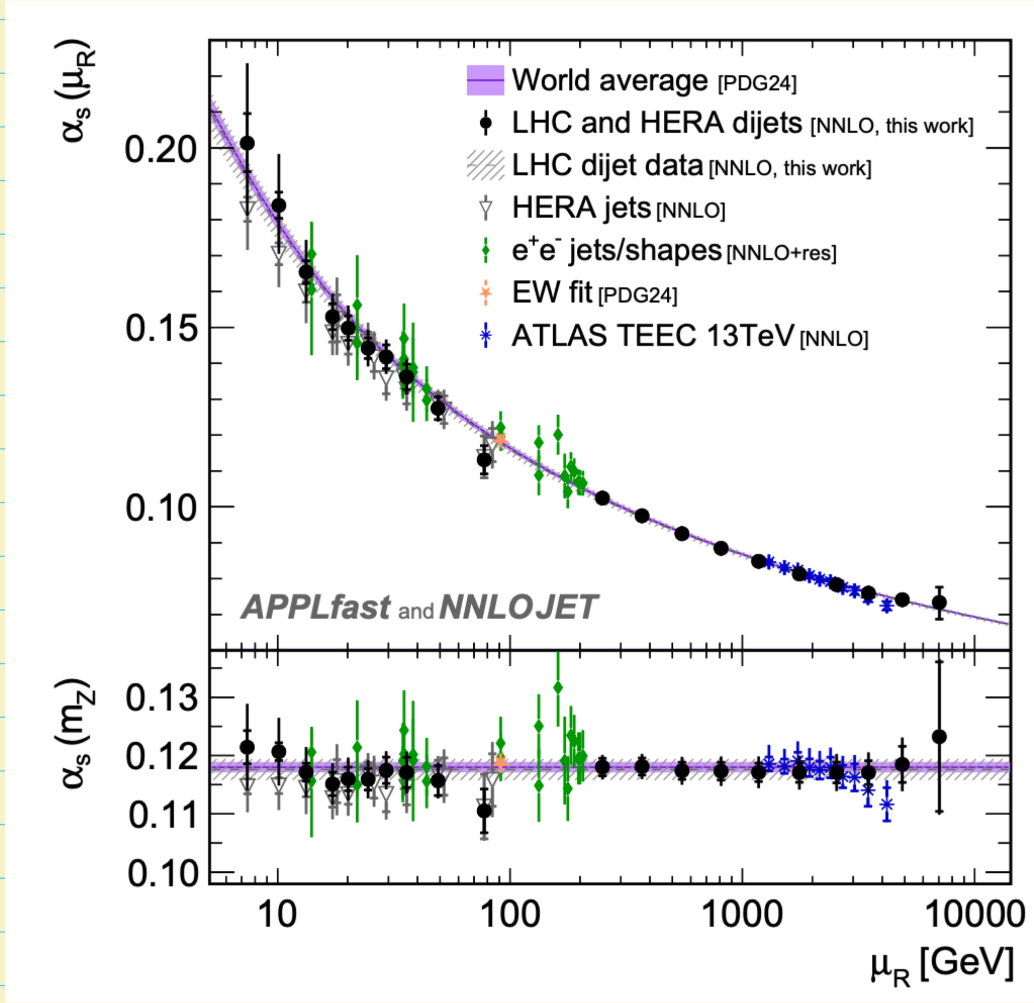
$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_F \quad > 0$$

$$\boxed{\frac{d\alpha_s(\mu)}{d\ln\mu} < 0 \;\; !}$$

asymptotic freedom :

The strength of α_s becomes smaller at shorter distance or equivalently, large scales.





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hatched area fit $\alpha_s(m_Z)$ + RGE

in good agreement with data

$$\alpha_s(\mu) \simeq \frac{\alpha_s(\mu_0)}{1+\frac{\alpha_s(\mu_0)}{2\pi}\beta_0 \ln\frac{\mu}{\mu_0}}$$

$$= \frac{2\pi}{\beta_0}\frac{1}{\ln \mu/\Lambda_{QCD}}$$

where Λ_{QCD} is defined as

$$1+\frac{\alpha_s(\mu_0)}{2\pi}\beta_0 \ln\frac{\Lambda_{QCD}}{\mu_0} = 0$$

or

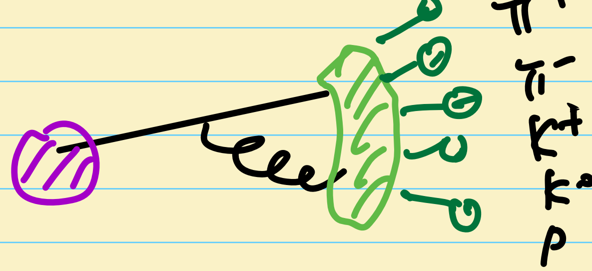
$$\Lambda_{QCD} \sim \mu_0 \exp\left[-\frac{2\pi}{\alpha_s(\mu_0)}\frac{1}{\beta_0}\right]$$

$$\sim 300\,\mathrm{MeV}$$

justifies pQCD at $Q >>> 1\,GeV$.
(gluons & quarks)

becomes non-perturbative
at scales $\sim 1\,GeV$ (hadrons)

However, experimentally what
we eventually observed are always
hadrons. Therefore we are always
probe scales $\lesssim 1\,GeV$.



How can pQCD work?

How are the calculations with quarks & gluons related to hadrons?

One answer is the parton-hadron duality. We will high-light the idea/assumption now