An Introduction to Perturbative QCD

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Following Jian's Lecture

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暗物质与新物理暑期学校

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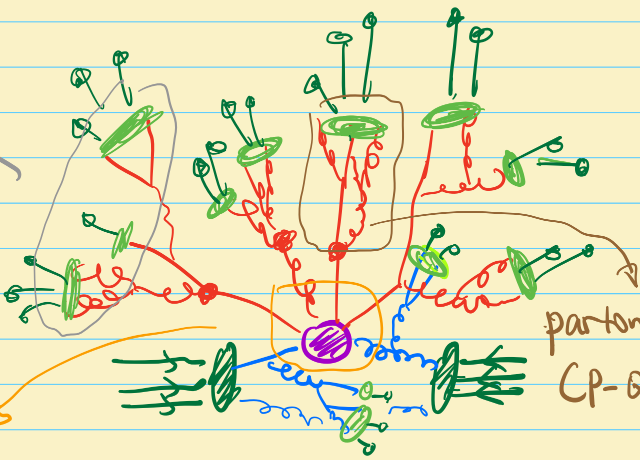
• Why (Perturbative) QCD ?

- $\alpha \sim \alpha_W \sim 10^{-2} <<< \alpha_s(M_W) \sim 0.1$

$\Rightarrow$ More QCD activities at a Collider.

hadronization by non-perturbative QCD



parton shower (P-QCD)

hard Interaction by perturbative QCD

a typical event @ LHC

$$\sigma = \hat{\sigma}_{ij}^{l\cdots}(\mu)\cdot F_{i/p} F_{j/p}(\mu)\cdot D_l(\mu)$$

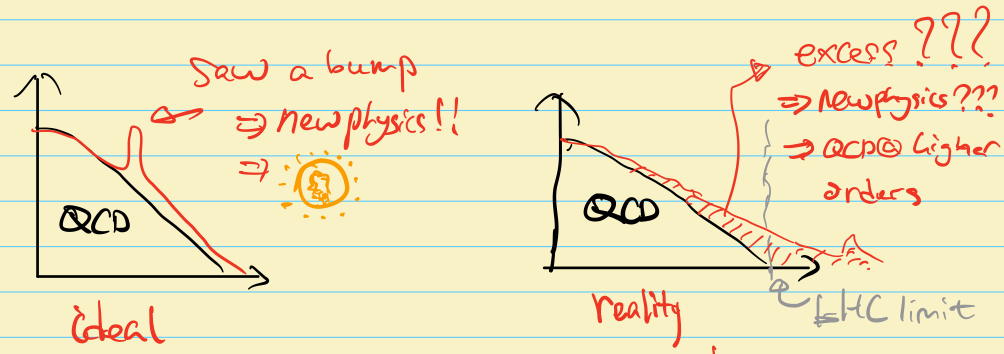
$\gtrsim \mathcal{O}(10\sim100\,\mathrm{GeV})$ $\hat{\sigma}(\mu)$

Possible other scales introduced by kinematics or cuts

evolution, resummation, Parton Shower deal with log of scales

$\sim \mathcal{O}(\Lambda_{QCD})$ $F(\mu\approx\Lambda_{QCD})$, $D(\mu\approx\Lambda_{QCD})$

- dominant background to new physics searches,



New physics ~ Data − QCD prediction.

- Fundaments to the Monte-Carlo tools

e.g. Madgraph. pythia. Sherpa. . . .

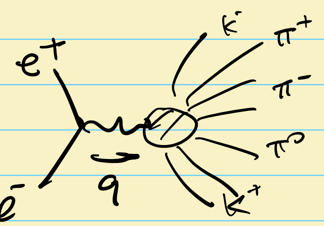
- Outline

- thrust as an example
  - NLO calculation
  - infrared safe
  - break-down of fixed order

- Infrared behaviour in QCD / QED
  - Coherent branching
  - DL Resummation

⊘ Thrust in e^+e^--annihilation

Consider



Measurement
- Incl. (1)
- trust (see later)

} scalar for simplicity

$$\sigma = \underbrace{\frac{1}{4}}_{\text{spin ave.}} \underbrace{\frac{1}{2s}}_{\text{Flux}} \int \underbrace{d\Phi_N}_{\text{Phase space of N final particles.}} L_{\mu\nu}(l,\bar{l}) \frac{1}{s^2} H^{\mu\nu}(q,\Phi_N)\, \Theta(\Phi_N)$$

scalar

$$\longrightarrow \frac{1}{4}\frac{1}{2s^3} L_{\mu\nu}(l,\bar{l}) \int d\Phi_N \left(g_{\mu\nu} - \cancel{\frac{q^\mu q^\nu}{q^2}}\right) H(q,\Phi_N)\, \Theta(\Phi_N)$$

gauge

$$= \frac{1}{4}\frac{1}{2s^3} L^\mu_\mu(l\bar{l}) \int d\Phi_N\, H(q,\Phi_N)\, \Theta(\Phi_N)$$

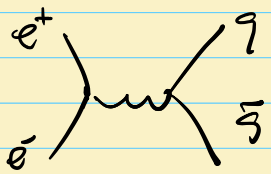
where

$$H = \frac{1}{d-1}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) H^{\mu\nu}$$

$$L^\mu_\mu = 4(2l\bar{l} - d\, l\bar{l}) = -2(d-2)s$$

For large $s = Q^2 \gg \Lambda_{QCD}$, it is well modeled by perturbative calculation

$$H^{\mu\nu}_{(1)} = 4\left(\hat{q}^\mu \bar{q}^\nu + \hat{q}^\nu \bar{q}^\mu - g^{\mu\nu}\, \hat{q}\cdot\bar{q}\right)$$



LO, $H^{(0)} = -\frac{1}{d-1}\, 2(d-2)s$

$$= \sum_f Q_f^2 N_c e^4 \, \frac{1}{4}\, \frac{1}{2s^3}\, \frac{[4(1-\epsilon)s]^2}{d-1} \int d\Phi_2^{\circledast}\, \Theta(\Phi_2) = 1$$

$$= \sum_f Q_f^2 N_c e^4 \, \frac{1}{4}\, \frac{1}{2s}\, \frac{16(1-\epsilon)^2}{3-2\epsilon}\, (2\pi)\, \frac{1}{4}\, \frac{\Omega_{d-2}}{(2\pi)^{d-1}}\, s^{-\epsilon} \int_0^1 dx\, x^{-\epsilon}(1-x)^{-\epsilon}$$

where $x \equiv \frac{-t}{s}$, $\Omega_{d-2} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$ is the solid angle

$$\Rightarrow \sigma_0 = \sum_f Q_f^2 N_c \frac{4\pi\alpha^2}{s}\, \frac{1-\epsilon}{3-2\epsilon}\, \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}$$

$$= \sum_f Q_f^2 N_c \frac{4\pi\alpha^2}{3s} + \mathcal{O}(\epsilon)$$

(*)

$$\int d\Phi_2 = \int \frac{d^d q}{(2\pi)^{d-1}} \delta(q^2) \frac{d^d \bar{q}}{(2\pi)^{d-1}} \delta(\bar{q}^2) (2\pi)^d \delta^{(d)}(Q-q-\bar{q})$$

$$= (2\pi) \int \frac{d^d q}{(2\pi)^{d-1}} \delta(q^2) \delta\left((Q-q)^2\right)$$

$$(Q-q)^2 = Q^2 - 2Q\cdot q = Q^2 - 2\ell\cdot q - 2\bar{\ell}\cdot q = s+u+t$$

$$= (2\pi) \int \frac{d^d q}{(2\pi)^{d-1}} \delta(q^2) \delta(s+u+t)$$

let $$q^\mu = \frac{2q\cdot\bar{\ell}}{2\ell\cdot\bar{\ell}} \ell^\mu + \frac{2q\cdot\ell}{2\ell\cdot\bar{\ell}} \bar{\ell}^\mu + q_\perp^\mu$$

$$= -\frac{u}{s}\ell^\mu + \frac{-t}{s}\bar{\ell}^\mu + q_\perp^\mu$$

Hence $$q^2 = \frac{ut}{s} - \vec{q}_\perp^{\,2}$$

and $$d^d q = \frac{1}{2} d\frac{(-u)}{s} d\frac{(-t)}{s} \, d^{d-2} q_\perp$$

$$= \frac{1}{4} d\frac{(-u)}{s} d\frac{(-t)}{s} (q_\perp^2)^{\frac{d-2}{2}} dq_\perp d\Omega_{d-2}$$

$$\Rightarrow \int d\Phi_2 = \frac{(2\pi)}{(2\pi)^{d-1}} \frac{1}{4} d\Omega_{d-2} \, s^{-\epsilon} \int_0^1 dx \, x^{-\epsilon} (1-x)^{-\epsilon}$$

For NLO, we have both virtual and real corrections

Virtual:

+C.T.

scaleless $\frac{1}{\cancel{\epsilon_{UV}}} - \frac{1}{\epsilon_{IR}}$

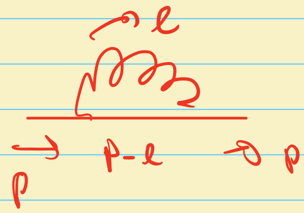
renorm.

$$\boxed{} = M_0^{\mu}\,\frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \underbrace{\left(-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\right.}_{\text{IR Poles}}\left.-2\right) \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}\, e^{\gamma_E \epsilon}$$

$\Rightarrow \sigma_{virt}$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} e^{\gamma_E \epsilon} \cos\pi\epsilon \left(-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}-2\right) \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{7}{6}\pi^2\right)$$

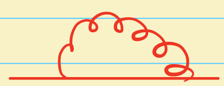


$p^2 = 0$

$$\propto \int d^d\ell \, \frac{1}{\ell^2 (p-\ell)^2} \sim \underbrace{(p^2)}_{\text{scaler}}{}^{\overbrace{\frac{d-4}{2}}^{\text{dim. analysis}}} = 0 = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$$

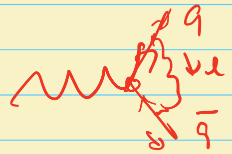
$\supset$

$$\int d^d\ell \, \frac{1}{\ell^4} \sim \text{UV div.}$$



$$= -\frac{\alpha_s}{2\pi}\left(\underbrace{\frac{1}{\epsilon}}_{UV} - \underbrace{\frac{1}{\epsilon}}_{IR}\right)$$

cancelled by counter term



$$[dl] \equiv \frac{d^d l}{(2\pi)^d}$$

$$= \int dl \; \bar{u}_i \, i g_s \gamma^\alpha t^a_{ik} \; i \frac{\not{q}+\not{l}}{(q+l)^2} \; ie\gamma^\mu \; (-i) \frac{\bar{\not{q}}-\not{l}}{(\bar{q}-l)^2} \; i g_s t^a_{kj} \gamma_\alpha v_j \; \frac{-i}{l^2}$$

$$= +ieg_s^2 (t^a t^a)_{ij} \int [dl] \boxed{\bar{u}_i \gamma^\alpha (\not{l}+\not{q}) \gamma^\mu (\not{l}-\bar{\not{q}}) \gamma_\alpha v_j} \boxed{\frac{1}{(q+l)^2} \frac{1}{(l-\bar{q})^2} \frac{1}{l^2}}$$

Now using the Feyn. Param. we find

$$\boxed{\frac{1}{(l+q)^2} \frac{1}{(l-\bar{q})^2} \frac{1}{l^2}} = \Gamma(3) \int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(L^2 - \alpha_2\alpha_3 s)^3}$$

$$\hookrightarrow \alpha_1 l^2 + \alpha_2 l^2 + 2\alpha_2 l\cdot q + \alpha_3 l^2 - 2\alpha_3 l\cdot\bar{q}$$

$$= l^2 + 2(\alpha_2 q - \alpha_3 \bar{q})\cdot l + (\alpha_2 q - \alpha_3 \bar{q})^2 + \alpha_2\alpha_3 s$$

$$= (\underbrace{l + \alpha_2 q - \alpha_3 \bar{q}}_{\equiv L})^2 + \alpha_2\alpha_3 s$$

We simplify the numerator

$$\boxed{} = \bar{u}_i \gamma^\alpha (\not{t} + \bar{\alpha}_2 \not{q} + \alpha_3 \not{q}) \gamma^\mu (\not{t} - \alpha_2 \not{q} - \bar{\alpha}_3 \not{q}) \gamma_\alpha v_j$$

$$= -2 \bar{u}_i (\not{t} - \cancelto{0}{\alpha_2 \not{q}} - \bar{\alpha}_3 \not{q}) \gamma^\mu (\not{t} + \bar{\alpha}_2 \not{q} + \cancelto{0}{\alpha_3 \not{q}}) v_j$$

$$+2\epsilon \bar{u}_i (\not{t} + \cancelto{0}{\bar{\alpha}_2 \not{q}} + \alpha_3 \not{q}) \gamma^\mu (\not{t} - \alpha_2 \not{q} - \cancelto{0}{\bar{\alpha}_3 \not{q}}) v_j$$

$\propto L^{2n}$

odd power integrated to zero

$$\longrightarrow -2 \bar{u}_i \not{t} \gamma^\mu \not{t} - \bar{\alpha}_3 \bar{\alpha}_2 \not{q} \gamma^\mu \not{q} v_j$$

$$+2\epsilon \bar{u}_i \not{t} \gamma^\mu \not{t} - \alpha_2 \alpha_3 \not{q} \gamma^\mu \not{q} v_j$$

$$= -2(1-\epsilon) \bar{u}_i \not{t} \gamma^\mu \not{t} v_j$$

$$+ (2\bar{\alpha}_3 \bar{\alpha}_2 - 2\epsilon \alpha_3 \alpha_2) \bar{u}_i \not{q} \gamma^\mu \not{q} v_j$$

$L^\mu L^\nu \to \frac{1}{d} L^2 g^{\mu\nu}$

$$\longrightarrow \boxed{\frac{(d-2)^2}{d} L^2 \bar{u}_i \gamma^\mu v_j + 2(\bar{\alpha}_2 \bar{\alpha}_3 - \epsilon \alpha_2 \alpha_3)(-s) \bar{u}_i \gamma^\mu v_j}$$

$d/2 = 2-\epsilon$

Now we use

GV

$$\int [dl] \frac{L^2}{(L^2 + \alpha_2 \alpha_3 s)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-\epsilon} (-s)^{-\epsilon} \quad (1)$$

$$\int [dl] \frac{1}{(l^2 + \alpha_2 \alpha_3 s)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-1-\epsilon} (-s)^{-1-\epsilon} \quad (2)$$

⇒

① ⇒

$$g_s^2 \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\cancel{\Gamma(3)}} (-s)^{-\epsilon} \frac{(d-2)^2}{d} \int_0^1 d\alpha_2 \int_0^{\bar{\alpha}_2} d\alpha_3 \, (\alpha_2\alpha_3)^{-\epsilon} \, \cancel{\Gamma(3)}$$

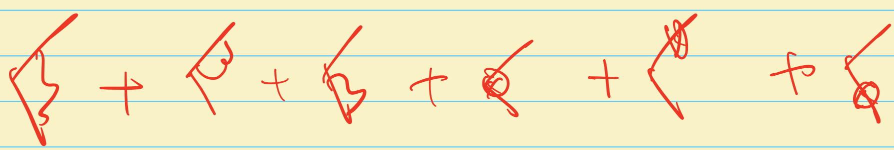
$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{+\epsilon} \left[\frac{1}{2\epsilon} + \frac{1}{2}\right]$$

↳ UV cancelled by counter term

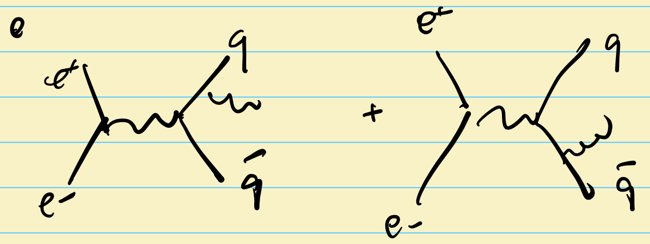
② ⇒

$$\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{+\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{2}{\epsilon} - \frac{9}{2} + \frac{\pi^2}{12}\right]$$

↳ coll ↳ IR



$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{3}{2}\frac{1}{\epsilon} - 4 + \frac{\pi^2}{12}\right]$$



NLO real

$$H^{(1)} = -C_F g_s^2 \, 8(1-\epsilon) \frac{1}{d-1}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

Where

$$y_1 = \frac{S_{12}}{Q^2}, \quad y_2 = \frac{S_{13}}{Q^2}, \quad y_3 = \frac{S_{23}}{Q^2}$$

and $S_{ij} = 2 q_i \cdot q_j$

$$d\Phi_3 = \frac{1}{(2\pi)^{2d-3}} \frac{1}{2^{d+1}} S^{d-3} \, d\Omega_{D-1} \, d\Omega_{D-2}$$

$$(y_1 y_2 y_3)^{-\epsilon} \int_0^1 dy_1 \, dy_2 \, dy_3 \, \delta(1-y_1-y_2-y_3)$$

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$$\times\ \Theta(\Phi_3)$$

Inclusive: $\Theta(\Phi_3) = 1$

let $y_1 = (1-z_1)$, $y_2 = z_1 z_2$, $y_3 = (1-z_2) z_1$

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

Jacobian

$$\times \int_0^1 dz_1\, dz_2\;\; \bar{z}_1^{-\epsilon}\, z_1^{-2\epsilon}\, z_2^{-\epsilon}\, \bar{z}_2^{-\epsilon}\;\; \underbrace{z_1}_{\text{Jacobian}}$$

$$\times \left\{ \frac{2}{z_1^2 z_2 \bar{z}_2} + \frac{-2+z_1\bar{z}_2-\epsilon z_1\bar{z}_2}{z_1 z_2} + \frac{-2+z_1 z_2-\epsilon z_1 z_2}{z_1 \bar{z}_2} - 2\epsilon \right\}$$

$$\Rightarrow \sigma_{real} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{7}{6}\pi^2\right)$$

$$\sigma_{virt.} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2\right)$$

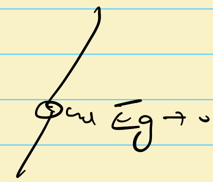
$$\sigma = \sigma_0 \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2} \right\}$$

- all IR poles cancelled, KLN Theorem
- Virt & Real almost completely cancelled with each other : $V + R \sim 1 \leftarrow$ Unity
- Origin of the IR poles :

all IR poles from the soft & coll. limit

1 $z_1 \to 0. \Rightarrow y_2 \,\&\, y_3 \to 0$

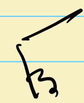
$\Rightarrow p_3 \to 0$ soft



2. z_2 or $\bar{z}_2 \to 0 \Rightarrow y_2$ or $y_3 \to 0$

$\Rightarrow p_3 \cdot p_1 \to 0,\ p_3 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_1$

or $p_3 \cdot p_2 \to 0,\ p_2 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_2$



$z_1 \to 0$

$$\int_0^1 dz_1 dz_2\, z_1^{-2\epsilon} z_2^{-\epsilon} \bar{z}_2^{-\epsilon} \times \left\{ \frac{2}{z_1 z_2 \bar{z}_2} \right\} = \frac{2}{\epsilon^2} + \cdots$$

soft collinear soft+coll.

$z_2 \to 0$

$$\int_0^1 dz_1 dz_2\, \bar{z}_1^{-\epsilon} z_1^{-2\epsilon} z_2^{-\epsilon}$$

$$\times \left\{ \cancel{\frac{2}{z_1 z_2 \bar{z}_2}} + \frac{-2 + z_1 - \epsilon z_1}{z_2} \right\} = \frac{3}{2} \frac{1}{\epsilon}$$

subtract out to avoid double counting

does reproduce all poles

every order you will have

n-emission

$$\Delta^{(n)}_{\text{real}} \propto \left(\frac{\alpha_s}{2\pi}\right)^n \left(\frac{\#_{2n}}{\epsilon^{2n}} + \cdots + \frac{\#_1}{\epsilon} + \text{finite terms} \right)$$

n soft + coll.

cancel against the IR poles in the virtual. for inclusive processes

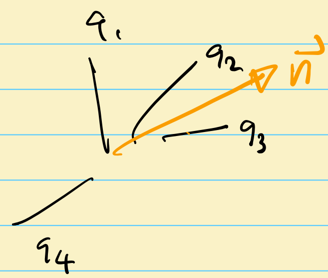
for exclusive processes.

e.g. $pp \to \cdots$

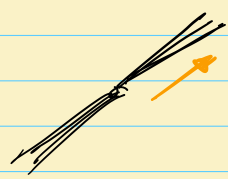
$e^+e^- \to \text{hadron} + \cdots$

remaining poles to be absorbed into PDFs or FFs.

thrust: $T = \max_{\vec{n}} \dfrac{\sum_i |\vec{q}_i \cdot \vec{n}|}{\sum_i |\vec{q}_i|}$

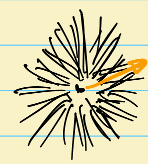


massless case $\sum_i |\vec{q}_i| = Q$



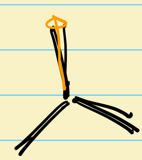
$T \to 1$

pencil-like



$T \to 1/2$

Perfect Isotropic

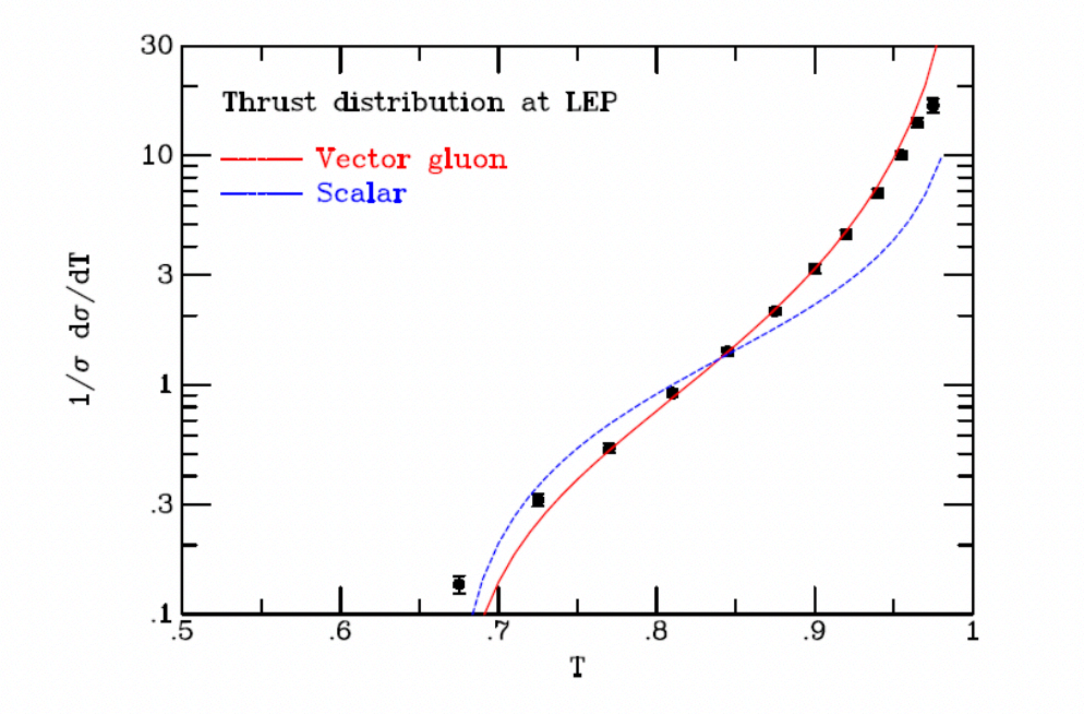


$T \to 2/3$

Mercedes-type

measure the shape of an event

as $T \to 0$, only soft or coll. radiations are allowed



Used for discovery of gluon

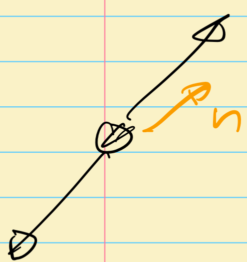
+ gluon spin determination

+ precision determination of α_s

restrictions on the phase space:

$$\Theta(\Phi_2) = \delta\left(T - \max_{\vec{n}} \frac{|\vec{q}_1\cdot\vec{n}| + |\vec{q}_2\cdot\vec{n}|}{Q}\right)$$

$$= \delta\left(T - \max_{\vec{n}} \frac{|\vec{q}_1\cdot\vec{n}| + |-\vec{q}_1\cdot\vec{n}|}{Q}\right)$$



$$= \delta\left(T - \max_{\vec{n}} \frac{Q/2\cos\theta + Q/2\cos\theta}{Q}\right)$$

$$= \delta\left(T - \max_{\vec{n}} \cos\theta\right)$$

$$= \delta(1-T) \quad \text{always.}$$

No restrictions

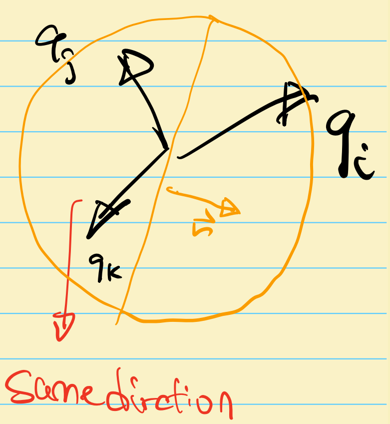
does not contribute to $T \neq 1$ case, which starts from 3-body final state

$$\sigma^{(0)} + \sigma^{(1)}_{virt} \rightarrow \left(\sigma^{(0)} + \sigma^{(1)}_{virt}\right)\delta(1-T)$$

$$\Theta(\bar{E}_3) = \delta\left(T - \max_{\vec{n}} \frac{|\vec{q}_1\cdot\vec{n}| + |\vec{q}_2\cdot\vec{n}| + |\vec{q}_3\cdot\vec{n}|}{Q}\right)$$

too Complicated for analytic calculation

- for an arbitary given $\vec{n}$, we can separate the phase space to 2 hemispheres, with one covers only one particle q_i



- the max of $\vec{n}$ for such config. is given by

$$\max \frac{|\vec{q}_i\cdot\vec{n}|}{Q} + \frac{|\vec{q}_j\cdot\vec{n}|}{Q} + \frac{|\vec{q}_k\cdot\vec{n}|}{Q}$$

$$= \max \frac{|\vec{q}_i\cdot\vec{n}|}{Q} + \left|\frac{\vec{q}_j\cdot\vec{n}}{Q} + \frac{\vec{q}_k\cdot\vec{n}}{Q}\right|$$

$$= \max \frac{|\vec{q}_i\cdot\vec{n}|}{Q} + \frac{|-\vec{q}_i\cdot\vec{n}|}{Q} = \max \frac{2|\vec{q}_i\cdot\vec{n}|}{Q} = \frac{2E_i}{Q} = x_i$$

$$= \frac{2q_i\cdot Q}{Q^2} = 1 - y_i$$

- We need to maximize all possible configs

$\Rightarrow T = \max(x_1, x_2, x_3) = \max(1-y_1, 1-y_2, 1-y_3)$

$\Rightarrow 1 - T = \min(y_1, y_2, y_3) \equiv \tau$

Which leads to the phase space restriction

$$\Theta(\Phi_3) = \sum_i \delta(\tau - y_i)\,\theta(y_i < y_j)\,\theta(y_i < y_k)$$
$$= \sum_i \delta(\tau - y_i)\,\theta(\tau < y_k < 1-2\tau)\,\theta(\tau < 1/3)$$

Plug into the real-emission cross section

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$
$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1-y_1-y_2-y_3)$$
$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$
$$\Theta(\Phi_3)$$

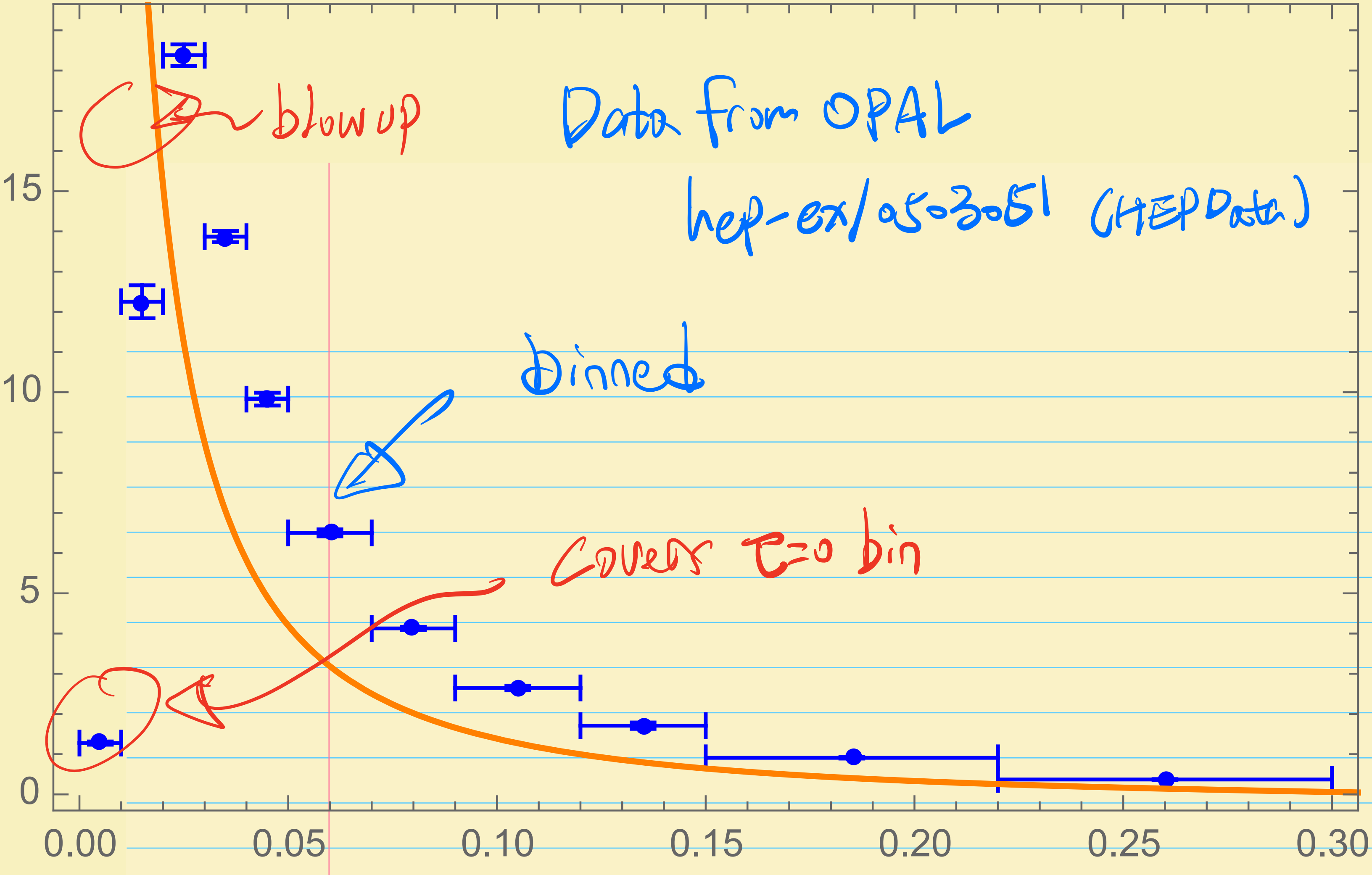
You can work out the integration with ε kept using mathematica

For simplicity, we set $\epsilon=0$ to find

$$\frac{d\sigma_{real}}{d\tau}$$

$$= \frac{\alpha_s}{2\pi} C_F \left\{ \frac{4}{\tau} \frac{\log\left(\frac{1-2\tau}{\tau}\right)}{1-\tau} - \frac{3(1+\tau)(1-3\tau)}{\tau} - 6\log\left(\frac{1-2\tau}{\tau}\right) \right\}$$

For $0 < \tau < 1/3$, otherwise vanishes.



Data from OPAL
hep-ex/0503051 (HEPData)
blowup
Binned
Covers τ=0 bin
15
10
5
0
0.00
0.05
0.10
0.15
0.20
0.25
0.30

We can slightly improve our result to cover the "$\tau=0$" bin.

Since we know $\sigma_{tot} = \sigma_0 \left(1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2}\right)$

hence $\int_0^{1/3} \frac{d\sigma}{d\tau} = \sigma_{tot}$

We assume our real calculation is only "Valid" for $\tau > \delta$, with $\delta \to 0$. Therefore

$$\frac{\sigma_{tot}}{\sigma_0}$$

an effective Virtual correction

$$= \int_\delta^{1/3} \frac{d\sigma_{real}}{d\tau} + \int_0^\delta \frac{d\sigma_{\text{"virt"}}}{d\tau}$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{5}{2} - \frac{\pi^2}{3} + 3\log[\delta] + 2\log^2[\delta] \right) + \Delta_{\text{"virt"}}$$

$$= 1 + \frac{3}{2} \frac{\alpha_s}{2\pi} C_F$$

* $\sigma_{real} + \sigma_{virt} \simeq 1$

this feature is useful for deriving Resummation

$$\Rightarrow \frac{``\sigma_{virt}"}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} C_F \left(-1 + \frac{\pi^2}{3} - 3\log[\delta] - 2\log^2[\delta] \right)$$

Hence the τ-distribution covers $\tau=0$ is given by

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \frac{1}{\sigma_0}\left[\lim_{\delta\to 0} \frac{d\sigma_{real}}{d\tau}\theta(\tau-\delta) + \frac{\sigma_{virt}}{\delta}\theta(\delta-\tau) \right]$$

$$= \lim_{\delta\to 0} \frac{1}{\sigma_0}\frac{d\sigma_{real}}{d\tau}\theta(\tau-\delta) - \frac{1}{\delta}\theta(\delta-\tau)\left(3\log[\delta] + 2\log^2[\delta]\right)\frac{\alpha_s}{2\pi}C_F$$

$$\left\{ 1 + \frac{\alpha_s}{2\pi}C_F\left(-1+\frac{\pi^2}{3}\right)\right\}\delta(\tau)$$

now we introduce the "+"-precription

$$\left(\frac{\log^{n-1} z}{z}\right)_+ = \lim_{\delta\to 0} \theta(z-\delta)\frac{\log z}{z} + \frac{1}{n+1}\log^n(\delta)\,\theta(\delta-z)$$

You can show that it is equivalent to what you are familiar with

$$\int_0^1 \left(\frac{\log^{n-1} z}{z}\right)_+ f(z)\, dz = \int_0^1 \frac{\log^{n-1} z}{z}\left(f(z)-f(0)\right) dz$$

by noting that

$$\sigma_{real} + \Delta_{\text{"virt"}} = \sigma_{tot}$$

$$\Rightarrow \Delta_{\text{"virt"}} = \sigma_{tot}\,\delta(1-z) - \int_0^1 \frac{d\sigma_{real}}{dz'} dz'\,\delta(1-z)$$

$$= \Delta_{tot}\,\delta(1-z) - \underbrace{\int_0 \frac{d\sigma^{reg.}_{real}}{dz'}}_{\text{finite as } z' \to 0} \delta(1-z) - \underbrace{\int_0 \frac{d\sigma^{sing.}_{real}}{dz'}}_{\text{div. as } z' \to 0} \delta(1-z)$$

combine with $\frac{d\sigma_{real}}{dz}$

$$\frac{d\sigma_{real}}{dz} - \int \frac{d\sigma^{sing}_{real}}{dz'} \delta(1-z) \to \text{plus prescription}$$

eventually we find

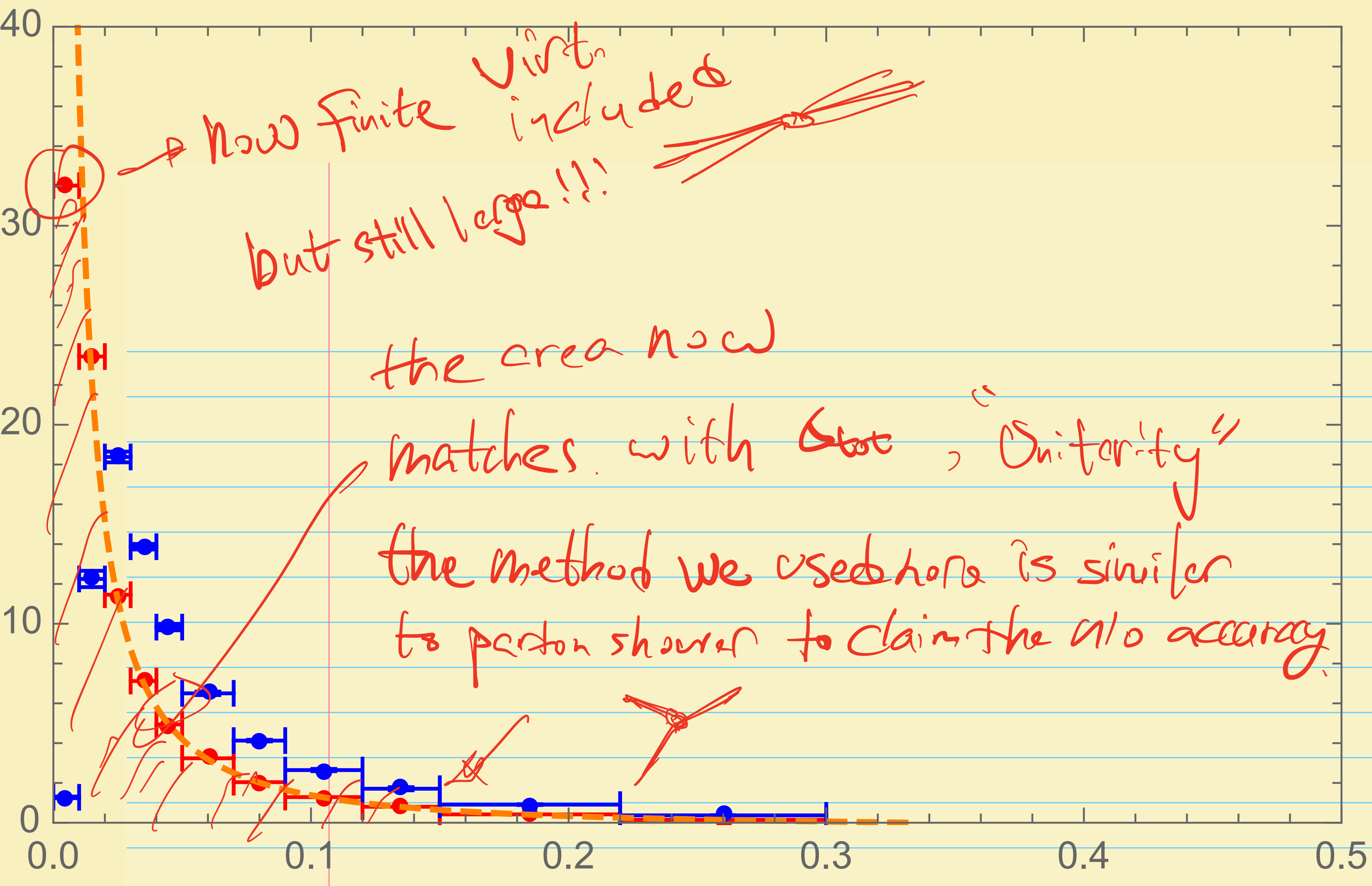
$$\boxed{\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \delta(\tau) + \frac{\alpha_s}{2\pi} C_F \Big\{ & -\frac{4}{1-\tau}\left(\frac{\log\tau}{\tau}\right)_+ - 3(1+\tau)(1-3\tau)\left[\frac{1}{\tau}\right]_+ \\ & + \left(-1+\pi^2/3\right)\delta(\tau) \\ & + \frac{4}{1-\tau}\frac{\log(1-2\tau)}{\tau} - 6\log\left(\frac{1-2\tau}{\tau}\right)\Big\}\end{aligned}}$$

The result is equivalent to keep ε in real do the calc. $\to$ Laurent expand in ε, + Virt.

We can thus have the cumulant cross-section

$$\boxed{\begin{aligned}\frac{\sigma(\tau<\delta)}{\sigma_0} &= 1+\frac{\alpha_s}{2\pi}C_F \\ &\Big\{-2L^2-3L-1-4\,\mathrm{Li}_2(-1+2\delta) \\ &\quad +6\delta\Big[\log\Big(\frac{\delta}{1-2\delta}\Big)+1\Big]+\frac{9}{2}\delta^2+3\log(1-2\delta) \\ &\quad -4\Big[\log(1-2\delta)\log(1-2\delta)-\log(1-\delta)\log\delta \\ &\qquad -\mathrm{Li}_2(\delta)+\mathrm{Li}_2(2\delta)\Big]\Big\}\end{aligned}}$$

For $\delta \leq 1/3$, $L \equiv \log\delta$



→ now finite virt. included

but still large!!!

the area now matches with ~~σ_{tot}~~, "Unitarity"

the method we used here is similar to parton shower to claim the nlo accuracy.

- in the "$\tau=0$"-bin, there is an exact cancellation between real & Virt. poles

$$\frac{1}{\sigma_0}\frac{d\sigma_{virt}}{d\tau} = \frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{7}{6}\pi^2\right)\delta(1-\tau)$$

$$\frac{1}{\sigma_0}\frac{d\sigma_{real}}{d\tau} = \frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}\cdots\right)\delta(1-\tau)+\cdots$$

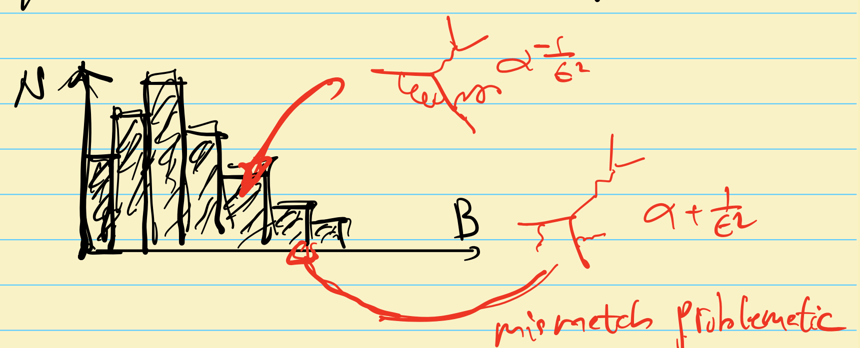
The cancellation ensures the predictive power of a F.O. calculation.

Not all observable has this feature !!)

recall that

- all poles come from the soft & coll. limit of the emitted partons,
- virt correction does not change multiplicities

$\Rightarrow$ to cancel poles, soft & coll. of real should fall into the same bin of virt.



IRC safe observables:

$$\Theta_N(p_1, p_2, p_3 \cdots p_i \cdots p_n) \xrightarrow[p_i \to 0]{\lim} \Theta_{N-1}(p_1, p_2, p_3 \cdots p_{i-1}, p_{i+1} \cdots p_N)$$

$$\Theta_N(p_1, p_2\, p_3 \cdots p_i \cdots p_j \cdots p_N) \xrightarrow[p_i \parallel p_j]{\lim} \Theta_{N-1}(p_1\, p_2 \cdots p_i{+}p_j \cdots p_{j-1}, p_{j+1} \cdots p_N)$$

You can check that thrust is IR safe.

IRC unsafe:

e.g. particle numbers

$$O_N(p_1 \cdots p_N) = N \xrightarrow[p_1 \to 0]{\lim} N \neq O_{N-1}(p_2 \cdots p_N) = N-1$$

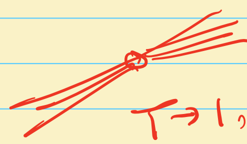
We need IRC safe observables to make F.O. predictive.

- L indicates the incomplete cancellation between real. & Virt.

as $\delta \to 1/3$, $G^{(1)} \to \frac{\alpha_s}{2\pi} C_F \, 3/2$, "Complete"

as. $\delta \to 0$, $L \to \infty$, recovers the divergence

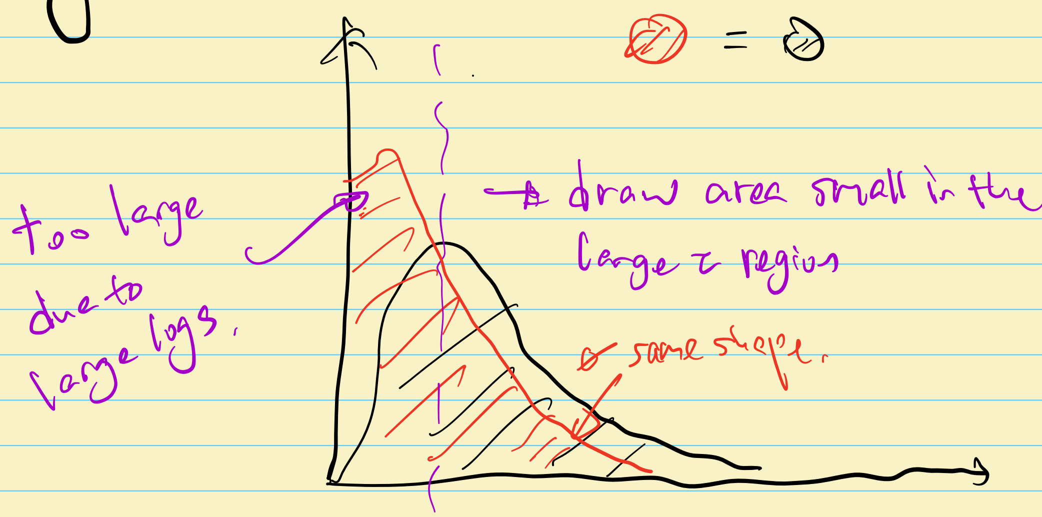
recall only soft and coll. radiations are allowed

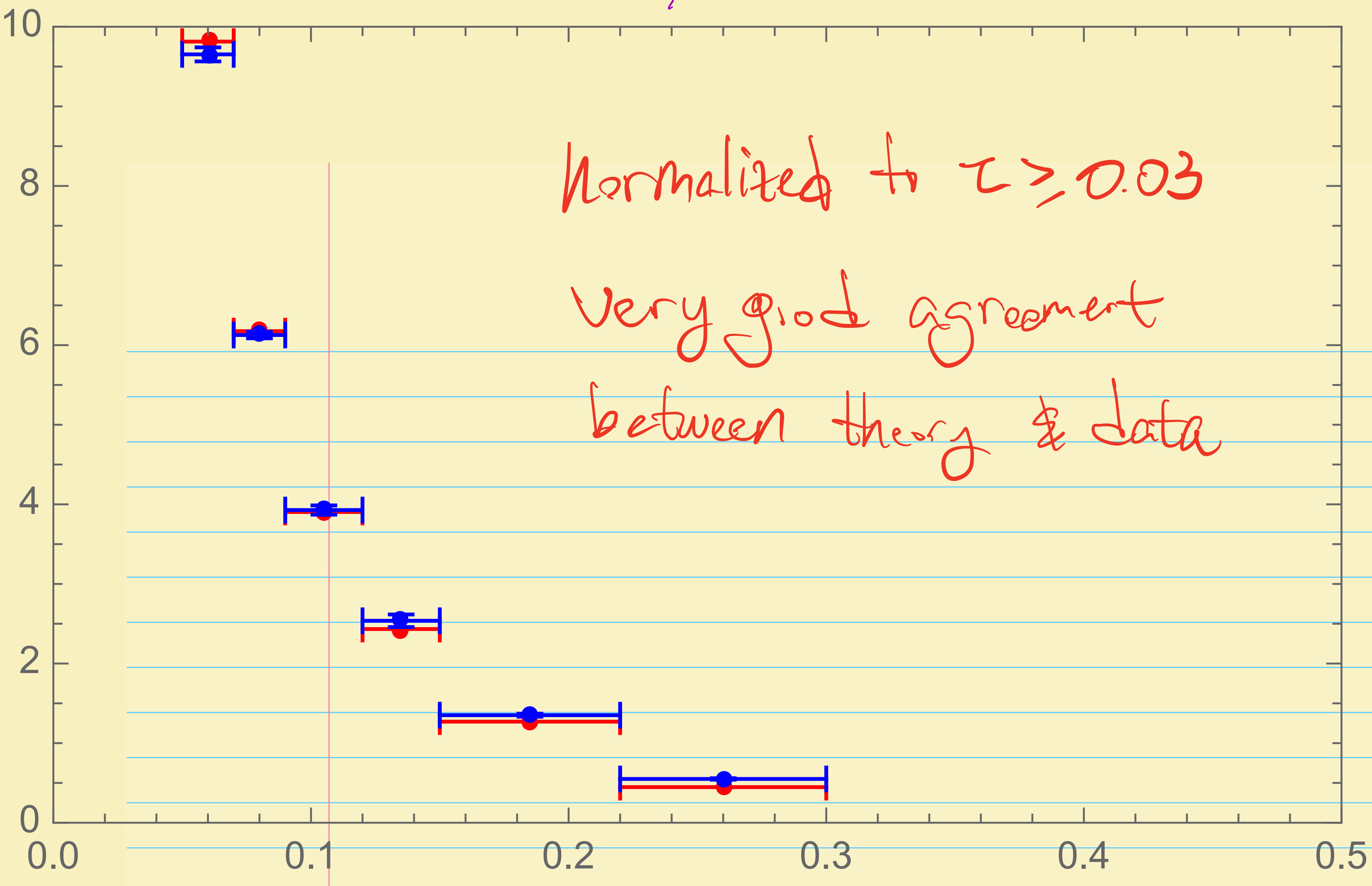


$T \to 1$, $\delta \to 0$

- Break down of F.O.

The entire $\frac{1}{\sigma}\frac{d\sigma}{d\tau}$ spectrum does not agree with data, all due to large logs in the small τ region





Normalized to $\tau \geq 0.03$
Very good agreement
between theory & data

Now we focus on the $\delta \ll 1$ region,

When $\delta \ll 1$, only soft & coll. radiations are allowed, and we have

$$\frac{\sigma(\delta)}{\sigma_0} = 1 - \frac{\alpha_s}{2\pi} C_F \left(\frac{4}{2} L^2 + 3 \cdot L + 1 - \pi^2/3 \right)$$

recall the poles

$$\frac{\sigma_{rad}}{\sigma_0} \sim \frac{\alpha_s}{2\pi} C_F \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} \cdots \right)$$

soft & collinear nature.

Indeed, recall $\ln = \int \frac{dx}{x}$

Hence, the logs in

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$\Theta(\Phi_3)$

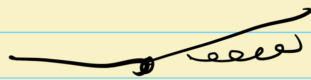
double log

single log.

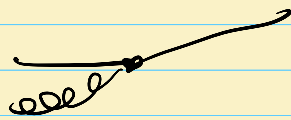
Comes from $y_2 \to 0$, $y_3 \to 0$,

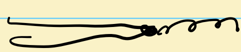
but $y_1 \to 0$ does not generate any logs

Pictorially:

$y_2 \to 0$  + soft g $\Rightarrow \tau = y_2$

recall $\tau = \min(y_i)$

$y_3 \to 0$  + soft g $\Rightarrow \tau = y_3$

$y_1 \to 0$  + soft quark $\Rightarrow \tau = y_1$

In this limit:

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 \cancel{dy_1}\, dy_2\, dy_3 \,(\cancel{y_1}^{\,1} y_2 y_3)^{-\epsilon}\, \cancel{\delta(1-y_1-y_2-y_3)}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \cancel{\frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon} \right\}$$

if keep only double log.

$\Theta(\tau_3)$

$\hookrightarrow \left[\delta(\tau-y_2)\,\Theta(\tau < y_3) + y_2 \leftrightarrow y_3 + \text{log suppressed}\right]$

$\epsilon \to 0$

$$= -\sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \; \frac{2\times 2}{\tau} \log \tau \quad + \text{ log suppressed}$$

$$\longrightarrow -\sigma_0 \frac{\alpha_s}{2\pi} C_F \; 4 \left(\frac{\log \tau}{\tau}\right)_+$$ by "Unitarity"

cumulative
$$\longrightarrow -\sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot 2 \ln^2 \delta$$ & one power from soft
one power from coll.

soft-collinear limit

reproduce the leading log structure

To all orders, we have structures

$$\sigma \sim 1$$

$$\text{NLO} \quad + \alpha_s L^2 \quad + \alpha_s L \quad + \#$$

$$\text{NNLO} \quad + \alpha_s^2 L^4 \quad + \alpha_s^2 L^3 \quad + \alpha_s^2 L^2 \quad + \alpha_s^2 L \quad + \#$$

$$\text{N}^3\text{LO} \quad + \alpha_s^3 L^6 \quad + \alpha_s^3 L^5 \quad + \alpha_s^3 L^4 \quad + \alpha_s^3 L^3 \quad + \alpha_s^3 L^2 \cdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

When $\alpha_s L^2 \sim 1$, we can not truncate the series at fixed α_s

since the $\alpha_s^n L^{2n}$ for $n > 1$ are equally important. We need to resum $\alpha_s L^2$ to all orders.

$\sigma \sim 1$

NLO $+ \alpha_s L^2 + \alpha_s L + \#$

NNLO $+ \alpha_s^2 L^4 + \alpha_s^2 L^3 + \alpha_s^2 L^2 + \alpha_s^2 L + \#$

N^3LO $+ \alpha_s^3 L^6 + \alpha_s^3 L^5 + \alpha_s^3 L^4 + \alpha_s^3 L^3 + \alpha_s^3 L^2 \cdots$

$\vdots \qquad \vdots$

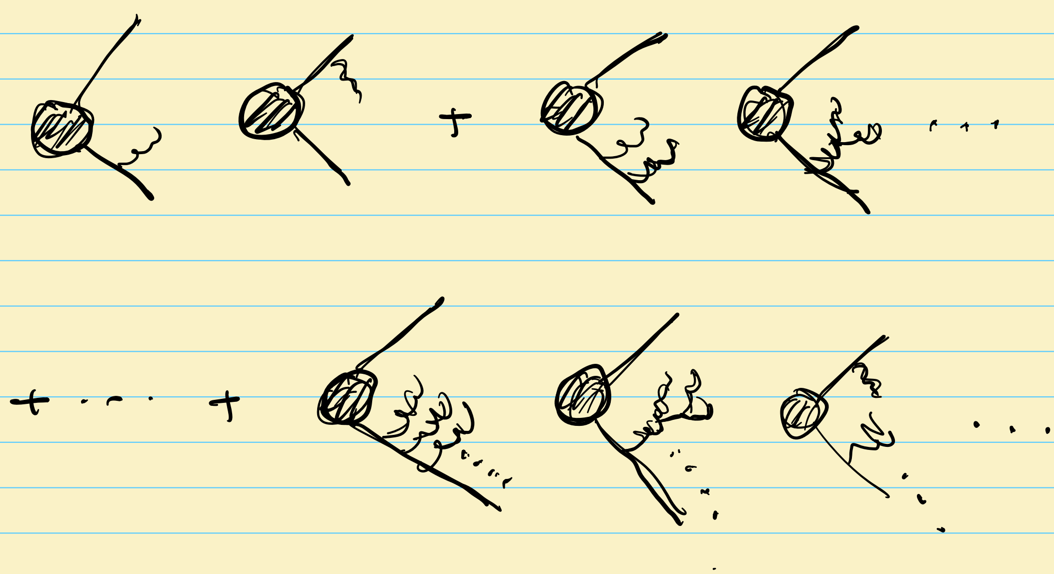
dominant logs, DL

For certain observables, the logs can be resummed in to an exponetiation form

$$\exp[\underbrace{L g_1(\alpha_s L)}_{\text{Leading log (LL)}} + \underbrace{g_2(\alpha_s L)}_{\text{next-to-leading Log (NLL)}} + \underbrace{\alpha_s g_3(\alpha_s L)}_{\text{NNLL}} \cdots]$$

Or equivalently.

We sum up the most singular behavior of the soft & coll. radiations to all orders



There are different Approaches to Resummation

- CSS formalism

  Collins, Soper, Sterman Nucl. Phys. B 250 (1985) 199-224
  hep-ph/0409313 (1989)

- Coherent branching algorithm — this lecture

  Catani et al. Nucl. Phys. B 327 (1989) 323-352
  Nucl. Phys. B407 (1993) 3-42

- Soft collinear effective theory

  Bauer et al. hep-ph/0202088, hep-ph/0109045
  hep-ph/0107001, hep-ph/0011336
  Beneke et al. hep-ph/0206152

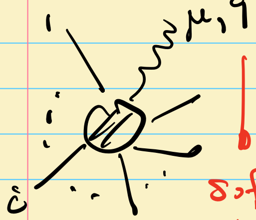
- CAESAR

  Banfi et al. hep-ph/0112156

All Approaches now have the ability to go beyond NLL.

- Infrared behaviour in QCD

→ a lesson from QED   gluon → photon



soft photon

$q \equiv |\vec{q}| \ll E_i$

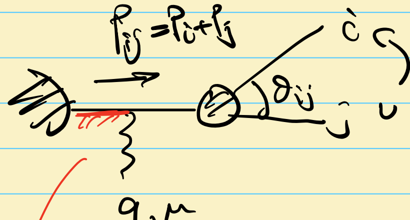
photon phase space:

$$\frac{d^3 q}{2q^0 (2\pi)^3} = \frac{q^2 dq \, d\Omega}{2q(2\pi)^3} = \boxed{\frac{1}{2(2\pi)^2} q \, dq \, d\cos\theta}$$

we are the interested in term $\propto \frac{1}{q^2}$ in the matrix element square.

- possible single emission

1. photon from internal line

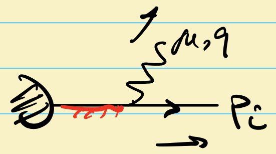


} hard, $\theta_{ij} \sim 1$, E_i, E_j large

$$\propto \frac{1}{(P_{ij}+q)^2} = \frac{1}{2P_{ij} \cdot q + P_{ij}^2} \xrightarrow{q \to 0} \frac{1}{P_{ij}^2}$$

regular as $q \to 0$

2. photon from external line



$$\bar u(p_i)(-ie\gamma^\mu)\, i\, \frac{\not p_i + \cancel{\not q}}{2p_i\cdot q} \cdots$$

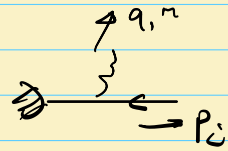
No $1/q$ enhancement

$$= \bar u(p_i)\, e\, \frac{\gamma^\mu \not p_i + \cancel{\not p_i \gamma^\mu}}{2p_i\cdot q}$$

$\to 0$, since $\not p_i u(p_i) = 0$

$$= \bar u(p_i)\, e\, \frac{p_i^\mu}{p_i\cdot q}$$

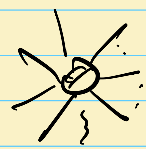
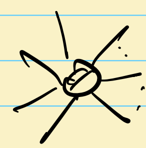
$\longrightarrow$ will give $\frac{1}{q^2}$ enhancement we need



$$\bar u(p_i)(-ie)\gamma^\mu\, i\, \frac{-\not p_i}{2p_i\cdot q}$$

$$= \bar u(p_i)(-e)\frac{p_i^\mu}{p_i\cdot q}$$

absorb "−" sign into the electric charge.

$\Rightarrow$  $\longrightarrow$ $\sum_i e_i \frac{p_i^\mu}{p_i\cdot q}\cdot$ 

q^a

- $q^\mu \mathcal{M}_\mu = \sum_i e_i \frac{q\cdot p_i}{p_i\cdot q} = \sum_i e_i = 0$

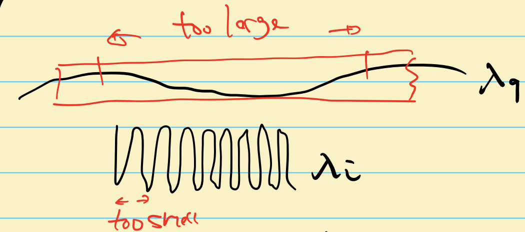
  charge conservation, gauge invariance.

- factorized soft current $J_i^\mu = e_i \frac{p_i^\mu}{p_i\cdot q}$

  independent of spin, Low theorem.

- a Factorization of the long & short distance physics.

$$\frac{1}{q} \sim \lambda_q \ggg \frac{1}{E_i} \sim \lambda_i$$



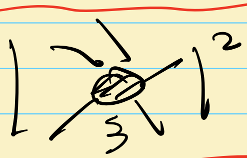
can not resolve the internal structure

Q: How about radiate a soft fermion?

$\Rightarrow$  soft leads to logs?

Now we square the Matrix element

$$\left| \quad \right|^2 \to -\sum_{i \neq j} e_i e_j \frac{p_i \cdot p_j}{p_i \cdot q \; p_j \cdot q} \left| \mathcal{M}_0(\{p_i\}) \right|^2$$



For single emission

define $$dw_1(q) = -d\Phi_1 \sum_{ij} e_i e_j \frac{p_i \cdot p_j}{p_i \cdot q \; p_j \cdot q}$$

& $$= -\sum_{ij} \frac{e_i e_j}{8\pi^2} \frac{p_i \cdot p_j}{p_i \cdot q \; p_j \cdot q} \cdot q\, dq\, d\cos\theta$$

$\frac{1}{2(2\pi)^2} q\, dq\, d\cos\theta$

Single photon emission prob.

Suppose we do a measurement v on the photon. that proposes the phase space restriction $\Theta_1(v)$ the prob. for the photon emission is

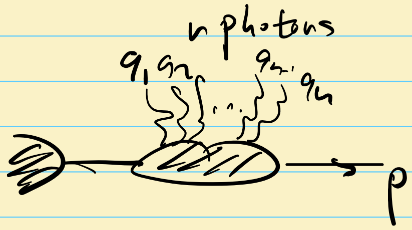
$W_1(v)$

$$= \int dv \left\{ \underbrace{\frac{dw_1(q)}{dv}\, \Theta_1(v)}_{\text{real}} - \underbrace{\int dv' \frac{dw_1(q)}{dv'}}_{\text{virt.}}\, \delta(v) \right\}$$

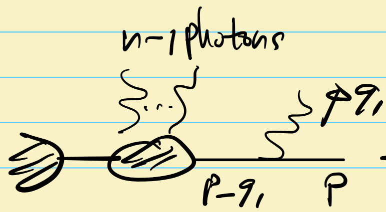
since $\int dv\,(\text{virt} + \text{real}) = 1$ + small corrections

$$\Rightarrow \quad \boxed{w_1(v) = \int dv \left[\Theta_1(v) - 1\right] \frac{dw_1}{dv}}$$

⊗ Multiple emissions :

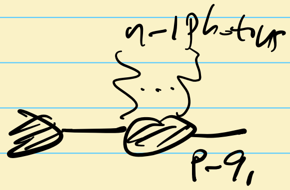


$\longrightarrow$



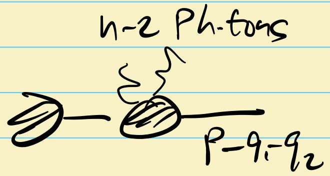
$+ \cdots$

$\longrightarrow$



$$e_i \frac{p^\mu}{p\cdot q_1} + \cdots$$

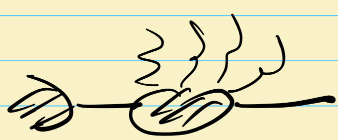
$\longrightarrow$



$$e_i \frac{p^\mu}{p\cdot q_1}\, e_i \frac{p^\nu}{p\cdot(q_1+q_2)} + \cdots$$

from $(p+q_1+q_2)^2$
$\sim 2p\cdot(q_1+q_2)$

$\Rightarrow$



$$= \frac{1}{n!} \sum_{\substack{\text{All}\\ \text{permutations}}} e_i \frac{p^{\mu_1}}{p_i\cdot q_1}\, e_i \frac{p^{\mu}}{p_i\cdot(q_1+q_2)} \cdots e_i \frac{p^{\mu_n}}{p_i\cdot(q_1+\cdots+q_n)}$$

$$= \frac{1}{n!} \prod_{j=1}^{n} e_i \frac{p^\mu}{p_i\cdot q_j}^{\,*} \longrightarrow \boxed{dW_n = \frac{1}{n!}\prod^{n} dW_1}$$

"independent" emission

$\Longrightarrow$

$\sum_n$ 

$\longrightarrow \sum_n \frac{1}{n!} \prod^n dw_i \, |M(\{p_i\})|^2$

- We note that the emission probability also depends on the phase space restriction which is not necessarily independent

$$\Theta_n(v) = \Theta_n(v(\{q_i\}))$$

- if the phase space restriction is also independent then

$$\sigma \longrightarrow \sigma_0 \exp[W_1(v)]$$

where

w_1 defined before including the "virtual".

$$W_1(v) = \int dv \left\{ \Theta_1(v) - 1 \right\} \frac{dw_1(q)}{dv}$$

You get the resummed form in QED for σ.

* $\Theta_c(v) = 1 \Rightarrow$ inclusive $\Rightarrow W_1(v) = 0$

$\Rightarrow \sigma \to \sigma_0$, Unitarity.

* $\Theta_c(v) = 0 \Rightarrow$ No emission

$\Rightarrow$

$$\sigma \to \sigma_0 \underbrace{\exp\left[- \underbrace{\int dv \frac{dW_1(q)}{dv}}_{-\infty \text{ "virtual only"}}\right]}_{\text{Sudakov Factor}} \to 0$$

$\Delta(\theta_{max}, 0)$

prob. of no emission below θ_{max}

* 2 photons,

$$\frac{p_i^\mu}{p_i \cdot q_1} \frac{p_i^\mu}{p_i \cdot (q_1 + q_2)} + \frac{p_i^\mu}{p_i \cdot q_2} \frac{p_i^\mu}{p_i \cdot (q_1 + q_2)}$$

$$= p_i^{\mu} p_i^{\nu} \frac{p_i \cdot q_2 + p_i \cdot q_1}{p_i \cdot q_1 \, p_i \cdot q_2 \, p_i \cdot (q_1 + q_2)} = \frac{p_i^{\mu}}{p_i \cdot q_1} \frac{p_i^{\nu}}{p_i \cdot q_2} \checkmark$$

suppose $\displaystyle\sum_{\text{perm.}} \frac{p_i^{\mu}}{p_i \cdot q_1} \frac{p_i^{\nu}}{p_i \cdot (q_1 + q_2)} \cdots \frac{p_i^{\mu}}{p_i \cdot (q_1 + \cdots q_{n-1})}$

$$= \frac{p_i^{\mu}}{p_i \cdot q_1} \cdots \frac{p_i^{\nu}}{p_i \cdot q_{n-1}}$$

then $\displaystyle\sum_{\text{perm}} \frac{p_i^{\mu}}{p_i \cdot q_1} \cdots \frac{p_i^{\nu}}{p_i \cdot (q_1 + \cdots q_{n-1})} \frac{p_i^{\mu}}{p_i \cdot (q_1 \cdots q_n)}$

$$= \sum_{\text{perm } j \neq n} \frac{p_i^{\nu}}{p_i \cdot q_1} \cdots \frac{}{p \cdots} \frac{p_i^{\mu}}{p_i \cdot (q_1 \cdots q_n)}$$

$$+ \sum_{\text{perm } j \neq 1} \frac{p_i^{\mu}}{p_i \cdot q_2} \cdots \frac{p_i^{\nu}}{p_i (q_2 + \cdots q_n)} \frac{p_i}{p_i \cdot (q_1 + \cdots q_n)}$$

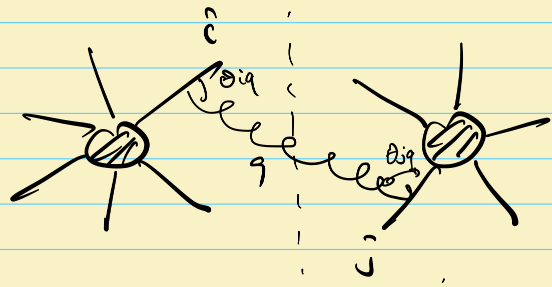
$+ \cdots$

$$= \left\{ \prod_{j \neq 1} \frac{p_i^{\mu}}{p_i \cdot q_j} + \prod_{j \neq 2} \frac{p_i^{\nu}}{p_i \cdot q_j} + \cdots + \prod_{j \neq n} \frac{p_i}{p_i \cdot q_j} \right\} \frac{p_i^{m}}{p_i \cdot (q_1 + \cdots + q_n)}$$

$$= \prod_j \left(\frac{p_i^{\nu}}{p_i \cdot q_j} (p_i \cdot q_1 + p_i \cdot q_2 + \cdots + p_i \cdot q_n) \frac{p_i^{m}}{p_i \cdot (q_1 + \cdots + q_n)} \right.$$

$$= \prod_j \frac{p_i^{m}}{p_i \cdot q_j}$$

⊘ QED coherence



$$\propto -\sum_{i\neq j} e_i e_j \frac{p_i\cdot p_j}{p_i\cdot q \; p_j\cdot q}$$

suppose a θ_{qk} is the smallest among all angles.

$$= -\sum_{i<j} e_i e_j \frac{2p_i\cdot p_j}{p_i\cdot q \; p_j\cdot q}$$

$$= -e_k \sum_{j\neq k} e_j \frac{2p_k\cdot p_j}{E_kE_q(1-\cos\theta_{qk})} \frac{1}{E_jE_q(1-\cos_{qj})} + \text{less singular}$$

$$= -\frac{e_k}{E_q^2}\frac{2}{1-\cos\theta_{qk}} \sum_{j\neq k} e_j \frac{p_k\cdot p_j}{E_jE_k(1-\cos_{qj})} + \text{less singular}$$

$$\simeq -\frac{2e_k}{E_q^2}\frac{1}{1-\cos\theta_{qk}} \underbrace{\sum_{j\neq k} e_j \frac{E_kE_j(1-\cos\theta_{kj})}{E_kE_j(1-\cos\theta_{kj})}}_{-e_k \text{ since charge cons.}} + \text{less singular}$$

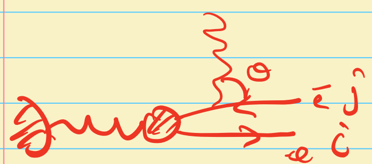
$$\simeq \frac{2 e_k^2}{E_q^2} \frac{1}{1-\cos\theta_{qk}} \quad (*)$$

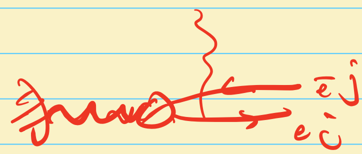
Here we used the coll. limit to derive the result but it can be derived as long as θ_{qk} is the smallest angle, no need for $\theta_{qk} \to 0$. See below.

Now we integrate over the photon phase space to find

$$dw_1(q) = \frac{\alpha}{\pi} \underbrace{\sum_k e_k^2 \frac{dq}{q} \frac{d\theta_{qk}^2}{\theta_{qk}^2}}_{\text{from a single independent emitter}} \underbrace{\Theta(\theta_{max} - \theta_{qk})}_{\text{QED Coherence due to destructive interference}}$$

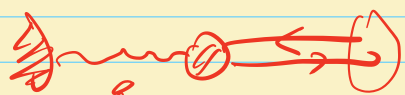
Here $\theta_{max} = \min\{\theta_{ij}\}$





$\theta_{ee} \ll \theta$

$p_j \parallel p_i$

$\Longrightarrow$ 

e

doesn't carry charge

see the overall charge $= 0$

hence the wide angle emission is suppressed

$$(-e)\frac{p_j^\mu}{p_j\cdot q} + e\frac{p_i^\mu}{p_i\cdot q} = (-e+e)\frac{p^\mu}{p\cdot q} = 0$$

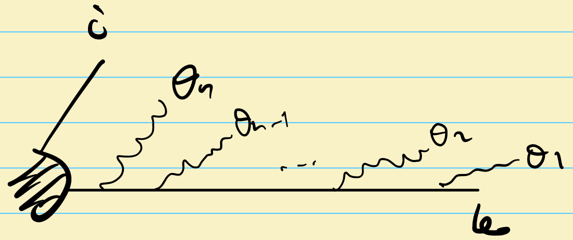
Therefore we find

$$\sigma = \sigma_0 \exp\left[\int dv \{\theta_1(v) - 1\} \frac{dW_1}{dv} \right]$$

$$\simeq \sigma_0 \prod_k \exp\left\{ \underbrace{\frac{\alpha}{\pi} e_k^2 \int dv \int \frac{dq}{q} \int^{\theta_{max}} \frac{d\theta_{qk}^2}{\theta_{qk}^2}}_{\text{Shower each leg}} \left[\theta_1(v) - 1\right] \right\}$$

$$= \sigma_0 \prod_k \sum_{n=0}^{\infty} \frac{1}{n!} \prod_n \frac{\alpha^n}{\pi^n} \left(e_k^2 \int \frac{dq}{q}\left[\theta_1(v) - 1\right]\right)^n \int^{\theta_{max}} \frac{d\theta_1^2}{\theta_1^2} \int^{\theta_{max}} \frac{d\theta_2^2}{\theta_2^2} \cdots \int^{\theta_{max}} \frac{d\theta_n^2}{\theta_n^2}$$

$$= \sigma_0 \prod_k \sum_{n=0}^{\infty} \prod_n \frac{\alpha^n}{\pi^n} \left(e_k^2 \int \frac{dq}{q}\left[\theta_1(v) - 1\right]\right)^n \underbrace{\int^{\theta_{max}} \frac{d\theta_n^2}{\theta_n^2} \int^{\theta_{n-1}} \frac{d\theta_{n-1}^2}{\theta_{n-1}^2} \int^{\theta_{n-2}} \frac{d\theta_{n-2}^2}{\theta_{n-2}^2} \cdots \int^{\theta_2} \frac{d\theta_1^2}{\theta_1^2}}_{\text{angular order}}$$



angular ordering to account for the interference

Very useful for Monte Carlo event-generator.

(*)

$$\frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} = S_{ij}^{(i)} + S_{ij}^{(j)}$$

$$S_{ij}^{(i)} = \frac{1}{2}\left\{ \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} + \frac{1}{E^2}\frac{1}{1-\cos\theta_{iq}} - \frac{1}{E^2}\frac{1}{1-\cos\theta_{jq}} \right\}$$

Where

$E_q(1,\ \sin\theta_{iq}\cos\phi_{iq}\ \ \sin\theta_{iq}\sin\phi_{iq},\ \ \cos\theta_{iq})$

$E_j(1,\ \sin\theta_{ij}\quad ,\quad 0\quad ,\quad \cos\theta_{ij})$

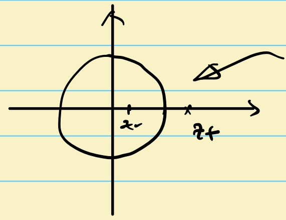
$$1-\cos\theta_{jq} = \underbrace{1-\cos\theta_{iq}\cos\theta_{ij}}_{c} - \underbrace{\sin\theta_{iq}\sin\theta_{ij}}_{s}\cos\phi_{iq}$$

Therefore

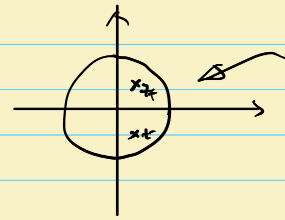
$$\int_0^{2\pi}\frac{d\phi_{iq}}{2\pi}\frac{1}{1-\cos\theta_{jq}} = \int_0^{2\pi}\frac{d\phi_{iq}}{2\pi}\frac{1}{c-s\cos\phi_{iq}} \qquad \text{let } z=e^{i\phi_{iq}}$$

$$= \frac{-i}{2\pi}\oint_{|z|=1}\frac{dz}{z}\frac{1}{c-s(z+1/z)} = \frac{i}{2\pi}\oint_{|z|=1} dz\,\frac{1}{sz^2-cz+s}$$

$$= \frac{i}{2\pi}\frac{1}{s}\oint_{|z|=1} dz\,\frac{1}{(z-z_+)(z-z_-)} \qquad \text{Where } z_\pm = \frac{1}{2}\frac{c}{s} \pm \frac{1}{2}\sqrt{\frac{c^2}{s^2}-1}$$



if $c/s > 1$



if $c/s < 1$

$\Rightarrow$ result $= -\frac{1}{s}\,\frac{1}{z - z_+}$ $\qquad\qquad \Rightarrow$ result $= 0$

$$= \frac{1}{\sqrt{c^2 - s^2}}$$

Therefore

$$\int \frac{d\phi_{iq}}{2\pi} S_{ij}^{(i)} = \frac{1}{2}\frac{1}{E^2}\int \frac{d\phi_{iq}}{2\pi}\left\{ \frac{1-\cos\theta_{ij}}{(1-\cos\theta_{iq})(1-\cos\theta_{jq})} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}} \right\}$$

$$= \frac{1}{2}\frac{1}{E^2}\left\{ \left[\frac{1-\cos\theta_{ij}}{1-\cos\theta_{iq}} - 1 \right] \frac{1}{\sqrt{c^2-s^2}} + \frac{1}{1-\cos\theta_{iq}} \right\}$$

$$= \frac{1}{2E^2}\left\{ \frac{\cos\theta_{iq} - \cos\theta_{ij}}{1-\cos\theta_{iq}}\,\frac{1}{|\cos\theta_{iq} - \cos\theta_{ij}|} + \frac{1}{1-\cos\theta_{iq}} \right\}$$

$$= \begin{cases} \frac{1}{E^2}\frac{1}{1-\cos\theta_{iq}} & \theta_{iq} < \theta_{ij} \\ 0 & \theta_{iq} > \theta_{ij} \end{cases}$$

$\Delta p \cdot \theta$ $\qquad \frac{\Delta p^2 \theta^2}{\Delta p} \sim \Delta p \cdot \theta \qquad \delta t \sim \frac{1}{\Delta \theta p}$

$\delta b \sim \frac{\theta_{ee}}{\ldots} \gg \lambda \sim \frac{1}{\Delta p \theta}$

$\theta_{ee} > \theta$

- summarize for QED

- $$W_1 = -4\pi\alpha \sum_{i \neq j} e_i e_j \int \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} \underset{q}{(\Theta_1(v) - 1)} \, dv \, d\Phi_q$$

virtual included

- if phase space factorize, then

$$d\sigma = d\sigma_0 \exp[W_1]$$

- Interference well approximated by time ordering

$$W_1 \to \frac{\alpha}{\pi} \sum_k e_k^2 \int \frac{dq}{q} \frac{d\theta^2}{\theta^2} \Theta(\theta_{max} - \theta_{qk}) (\Theta_1(v) - 1) dv$$

and

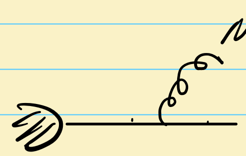
$$d\sigma = d\sigma_0 \exp[W_1]$$

$$= d\sigma_0 \prod_k \sum_{n=0}^{\infty} \prod_n \left(\frac{\alpha}{\pi} e_k^2 \int \frac{dq}{q} \{\Theta_1(v) - 1\} dv \right]^n \underbrace{\int_0^{\theta_{max}} \frac{d\theta_n^2}{\theta_n^2} \int^{\theta_n} \frac{d\theta_{n-1}^2}{\theta_{n-1}^2} \cdots \frac{d\theta_1^2}{\theta_1^2}}_{\text{angular ordering.}}$$

– soft emissions in QCD.

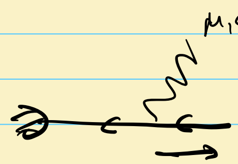
- single emission

Similar to the QED soft photon emission, we have the soft gluon radiated off a quark line.



$$= g_s t^a_{ij} \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_0^{j \ldots}$$

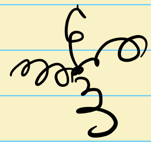
with e_i replaced by the color charge $g_s t^a_{ij}$ a matrix.

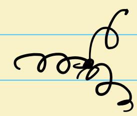


$$= g_s(-t^a_{ij}) \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_0^{j \ldots}$$

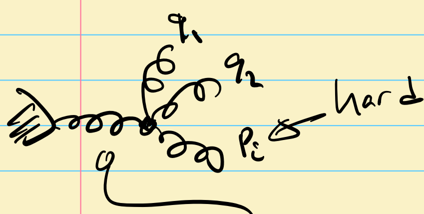
From anti-quark

More than that, since now the gluon carries charge



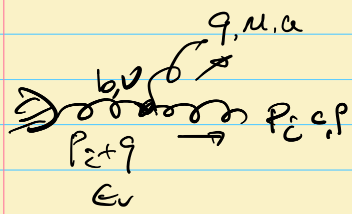


soft gluon can be emitted from hard gluons.



$$\frac{g_s^2}{(p_i+q_1+q_2)^2} = \frac{g_s^2}{\underbrace{2p_i(q_1+q_2)}_{E_1+E_2}+\underbrace{2q_1\cdot q_2}_{\cancel{E_1E_2}\to 0}} \sim \frac{g_s^2}{\underbrace{2p_i\cdot(q_1+q_2)}_{\text{not singular}}}$$

since from phase space we have $q_1 dq_1\ q_2 dq_2$, which returns suppressed results for either $q_1\to 0$ or $q_2\to 0$



$$\epsilon_\nu(p_i+q) \xrightarrow{q\ \text{soft}} \epsilon_\nu(p_i)$$

$$\propto \frac{1}{2p_i\cdot q}\, g_s f^{abc} \left\{ (+\overset{\text{gauge }0}{\cancel{2p_i}}+\underbrace{q}_{0})^\rho g^{\mu\nu} - (2p_i+\underbrace{q}_{0})^\mu g^{\nu\rho} + (\overset{\text{gauge }0}{\cancel{p_i}}-\underbrace{q}_{0})^\nu g^{\rho\mu} \right\} \epsilon_\rho \epsilon_\nu \mathcal{M}^b$$

non-singular

$$= -\frac{1}{2p_i\cdot q}\, g_s f^{abc}\; 2p_i^\mu\; \epsilon\cdot\epsilon\; \mathcal{M}_0^{b\ldots}$$

$$= g_s f^{abc} \frac{p_i}{p_i\cdot q} \mathcal{M}_0^{b\ldots}$$

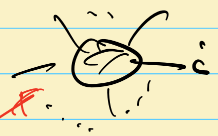
c.f. gluon from quark : $g_s t^a_{ij} \dfrac{p_i}{p_i\cdot q}$

- same eikonal factor, spin-independent
- with the color charge $g_s F^{abc} \equiv g_s f_c{}^{a}{}_{b}$

We can Unify the soft approximation for both quark and gluon emitter by introducing the color charge $\vec{T}_i^a$ where i is the i-th parton

$$
\begin{aligned}
\vec{T}_i^a &\to t^a \quad \text{if } i \text{ is a quark} \quad f \\
&\to \bar{t}^a = -t^a \quad \text{if } i \text{ is an anti-quark} \\
&\to f_{cb}^a \quad \text{if } i \text{ is a gluon.}
\end{aligned}
$$

Color algebra :



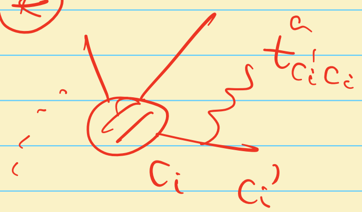
We note that $\sum_i \vec{T}_i^a = 0$ color charge conservation

cf. $\sum_i e_i = 0$ electric charge conservation

and if we define $\vec{T}_i^2 = \sum_a \vec{T}_i^a \vec{T}_i^a$, then

$$
\vec{T}_i^2 = \begin{cases} C_F & \text{for quark \& anti-quark} \\ C_A & \text{for gluon.} \end{cases}
$$

(*)



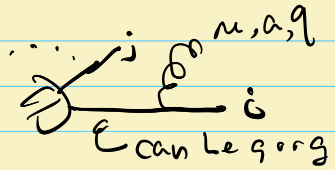
$$= g_s\, t^a_{c_i' c_i}\, \mathcal{M}^{\cdots c_i \cdots}$$

$$= g_s\, t^a_{c_i' c_i}\, |\cdots c_i \cdots\rangle$$

color operator acting on the color space, like a rotation

$$= g_s\, T^a_i\, |\cdots c_i \cdots\rangle$$

In this way



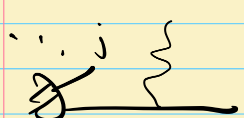
$$= g_s \sum_i \vec{T}_i^{\,a} \frac{p_i^\mu}{p_i\cdot q} \, |\mathcal{M}(\{p_i\})\rangle \qquad (*)$$

color charge operator acting on $|\mathcal{M}\rangle$

matrix element becomes a vector in the color space

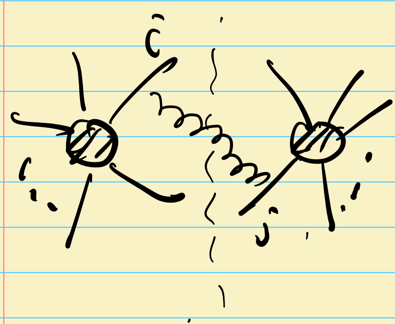
QCD is not truely factorized. there is color correlation remaining, since emitting gluon carries away color charge!!!

c.f. QED



$$= e\sum_i e_i \frac{p_i^\mu}{p_i\cdot q} \, \mathcal{M}_0(\{p_i\})$$

Now square the matrix element. We find

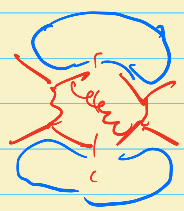

$$= -\sum_{i\neq j} g_s^2 \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} \langle M_0 | T_i\cdot T_j | M_0\rangle$$

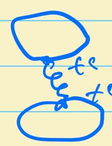
not entirely factorized

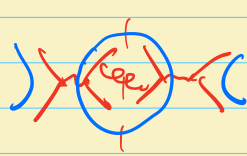
color correlation, depends on the color flow of the hard process

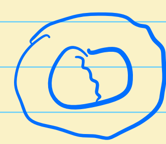
not proportional to $\langle M_0|M_0\rangle = |M_0|^2$

QCD:

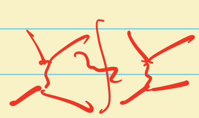
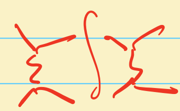

Color Flow


$= \mathrm{Tr}[t^a]\,\mathrm{Tr}[t^a] = 0$

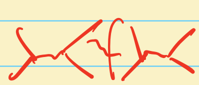

Color Flow


$= \mathrm{Tr}[t^a t^a]\,\mathrm{Tr}[1]$

QED:


$\sim e^2 \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q}$


exactly same!!


$\sim e^2 \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q}$


- multiple emissions

recall QED :

$$d\sigma \sim d\sigma_0 \exp[W_1]$$

$$W_1(v) = -4\pi\alpha \sum_{i\neq j} e_i e_j \int \frac{p_i \cdot p_j}{p_i \cdot q\, p_j \cdot q} (\Theta_1(v) - 1)\, dv\, d\Phi_q$$

direct exponentiation of 1-photo emission. (if the observable v is factorizable), since in QED,

$|M|^2 \sim |M_0|^2 e^2 \frac{p_i \cdot p_j}{p_i \cdot q\, p_j \cdot q}$ entirely factorizable.

How about QCD ?

Can we have

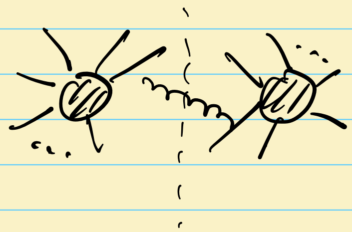
$$d\sigma \sim d\sigma_0 \exp[W_1]$$

$$W_1(v) = -4\pi\alpha \sum_{i\neq j} \vec{T}_i \cdot \vec{T}_j \int \frac{p_i \cdot p_j}{p_i \cdot q\, p_j \cdot q} (\Theta_1(v) - 1)\, dv\, d\Phi_q \quad ?$$

No. Can not factorize.!!

- color coherence

similar to QED, the Interference can be simplified in the Leading IR limit by angular ordering



$$= -g_s^2 \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} \langle M_0 | T_i \cdot T_j | M_0 \rangle$$

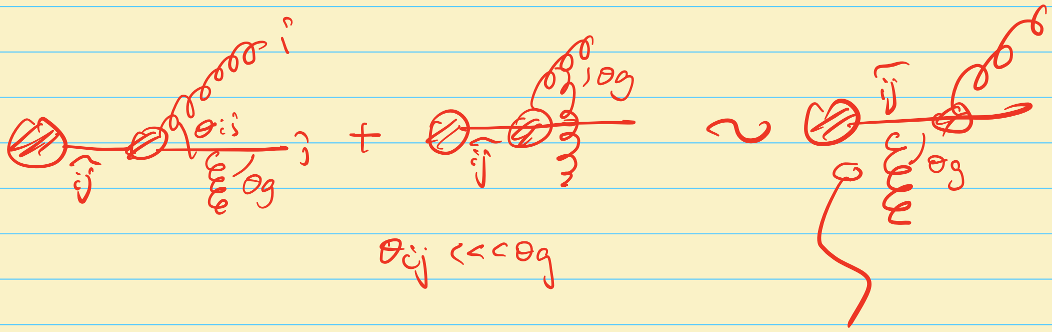
$$\simeq g_s^2 \sum_i T_i^2 \frac{2}{E^2} \frac{1}{\theta_{iq}^2} \Theta(\theta_{max} - \theta_{iq}) |M_0|^2$$

diagonal in the color space, c-#

does not "change" color

- QED-like Factorization
- angular ordering for Interference
- $T_i^2 = \begin{cases} C_F & \text{for } q \\ C_A & \text{for } g \end{cases}$

small angle radiation does not "change" the color charge.



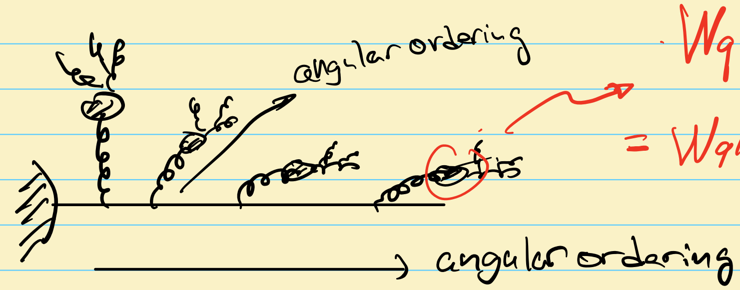
$\theta_{ij} \lll \theta_g$

$$T_i \frac{p^\mu}{p\cdot q} + T_j \frac{p^\mu}{p\cdot q} = \{\vec{T}_i + \vec{T}_j\} \frac{p^\mu}{p\cdot q} = T_{\tilde{ij}} \frac{p^\mu}{p\cdot q}$$

as if radiated from the hard emitter $\tilde{ij}$

$\boxed{\text{wide angle soft radiation only sees the overall color charge of the partons at smaller angles}}$

QCD muliple emission picture $\Rightarrow$ coherent branching



$W_q(V)\cdot M(p_1\ldots p_n, q)$

$= W_q(V)\,\frac{\alpha}{\pi}T_k^2\,\frac{dE_q}{E_q}\,\frac{d\theta_{qk}^2}{\theta_{qk}^2}\,\Theta_i(V)$

QCD

$$W_k(V) = 1 + \frac{1}{n!}\prod_{n=0}^{\infty}\left\{\frac{\alpha_s}{\pi}T_k^2\frac{dE_q}{E_q}\int^{\theta_{max}}\frac{d\theta_{qk}^2}{\theta_{qk}}\left\{\Theta_i(V(q))\,W_q(V)-1\right\}\right\}^n$$

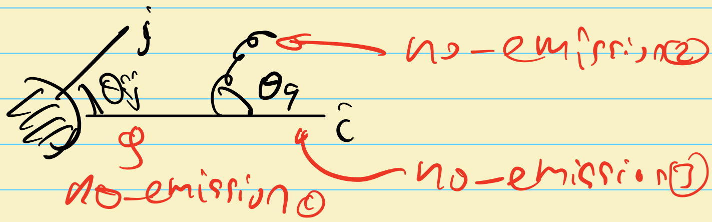
$$= \exp\left[\frac{\alpha_s}{\pi}T_k^2\int\frac{dE_q}{E_q}\int^{\theta_{max}}\frac{d\theta_{qk}^2}{\theta_{qk}^2}\left\{\Theta_i(V)\,W_q(V)-1\right\}\right]$$

- the soft gluon as the new emitter.
- small angle $\theta_{max}\sim\theta_{qk}$ now

$W(V) = \prod_k W_k(V)$, $\sigma = \sigma_0\cdot W(V)$

- the fundament to parton shower

e.g. Prob. for one-emission



$$\underbrace{W_i(v=0,\theta_{ij},\theta_g)}_{\text{Sudakov ①}}\underbrace{\frac{\alpha_s}{\pi}T_i^2\frac{dE_g}{E_g}}_{\text{1-emission prob.}}\frac{d\theta_g^2}{d\theta_g^2}\underset{②}{W_g(v=0,\theta_g,0)}\,\underset{③}{W_i(v=0,\theta_g,0)}$$

1-emission generated using Markov-chain

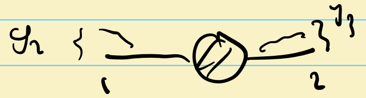
- can be improved to sub-leading logs

$$\frac{\alpha_s}{\pi}T_i^2\frac{dE}{E}\frac{d\theta^2}{\theta^2}\to\frac{\alpha_s}{2\pi}\frac{d\theta^2}{\theta^2}P_{a\to ij}(z)\,dz$$

← splitting function

- applied to analytic Resummation

DL resummation for the thrust.



we have two branchings, and we consider $y_2 \to 0$. $y_3 \to 0$ can be included in the same way

We first consider the 1-gluon emission

Recall that for the thrust $\tau \ll 1$

$$\Theta_1(v) = \Theta(\delta-\tau)\,\delta(\tau-y_2)\,\Theta(y_3-y_2) + 2\leftrightarrow 3$$

$$+ \cancel{1\leftrightarrow 2} \to 0 \rightsquigarrow \text{does not lead to LL}.$$

$$\simeq \Theta\Big(\delta - \underbrace{\frac{4E_1E_q}{4E^2}\frac{\theta^2}{2}}_{\tau}\Big)\,\Theta\Big(\frac{4EE_q}{4E^2}-\tau\Big)$$

$$= \Theta(\delta-\tau)\,\Theta\Big(\underbrace{\frac{E_q}{E}}_{z}-\tau\Big)$$

Therefore we have for one-gluon case:

$$\frac{dE_q}{E_q}\frac{d\theta_q^2}{\theta_q^2}\,\Theta_1(v) = \int^1 \frac{d\tau}{\tau}\int_\tau^1 \frac{dz}{z}\,\Theta(\delta-\tau)$$

For n-emissions we have

$$\Theta_n(v) = \Theta(\delta - \sum_i \tau_i) \prod_i^n \delta(\tau_i - y_{i\omega}) \Theta(y_{ri} - y_{2i})$$

$$= \Theta(\delta - \sum_i \tau_i) \prod_i^n (z_i - \tau)$$

Does not factorize!! Factorized

Hence we have for n-emission

$$\frac{dE_i}{E_i} \frac{d\theta_i^2}{\theta_i^2} \Theta_n(v)$$

$$= \left\{ \prod_{i=1}^n \int^1 \frac{d\tau_i}{\tau_i} \int_{\tau_i}^1 \frac{dz_i}{z_i} \right\} \Theta(\delta - \sum_i \tau_i)$$

To factorize $\Theta(\delta - \sum_i \tau_i)$, we note that

$$\Theta(\delta - \sum_i \tau_i) = \frac{1}{2\pi i} \int \frac{d\nu}{\nu} e^{\nu\delta} e^{-\sum_i \tau_i \nu}$$

to find

$$\frac{dE_i}{E_i}\,\frac{d\theta_i^2}{\theta_i^2}\,\Theta_n(v)$$

$$= \frac{1}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\delta} \prod_{i=1}^{n} \int^1 \frac{d\tau_i}{\tau_i}\int_{\tau_i}^1 \frac{dz_i}{z_i}\, \underbrace{e^{-\tau_i \nu}}_{\Theta_1(v)}$$

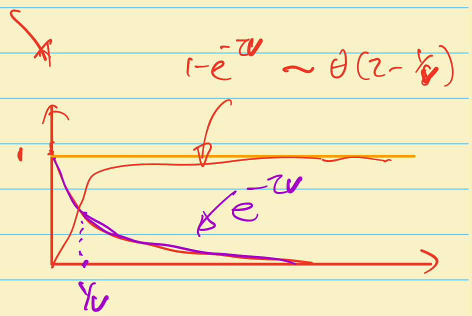
Now we plug it into $W(v)$

$$W(\delta) = \frac{1}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\delta} \exp\left[2\int^1 \frac{d\tau}{\tau}\int_\tau^1 \frac{dz}{z}\,\frac{\alpha_s}{\pi} C_F \left\{ e^{-\tau\nu} \underbrace{-1}_{\text{virtual}} \right\}\right]$$

We further use the fact that

$$e^{-\tau\delta} - 1 \simeq -\Theta(\tau - e^{-\gamma_E}\nu^{-1})$$

which holds up to NLL.



If we ignore the τ & z dependence in α_s which is sub-leading, we can have

$$W(\delta) = \frac{1}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\delta}\, e^{-\frac{\alpha_s}{\pi}C_F(\gamma_E+\log\nu)^2}$$

now we let $u = \nu\delta$ to have

$$W(\delta) = \frac{1}{2\pi i}\int du\, e^{u}\, e^{-\log u}\, e^{-\frac{\alpha_s}{\pi}C_F(\log u+\log e^{\gamma_E}\delta^{-1})^2}$$

$$= e^{-\frac{\alpha_s}{\pi}C_F\log^2 e^{\gamma_E}\delta^{-1}}\, \underbrace{\frac{1}{2\pi i}\int du\, e^{u}\, e^{-\log u}}_{\Theta(U)}\, e^{-\frac{\alpha_s}{\pi}C_F(\log^2 u+2\ldots)}$$

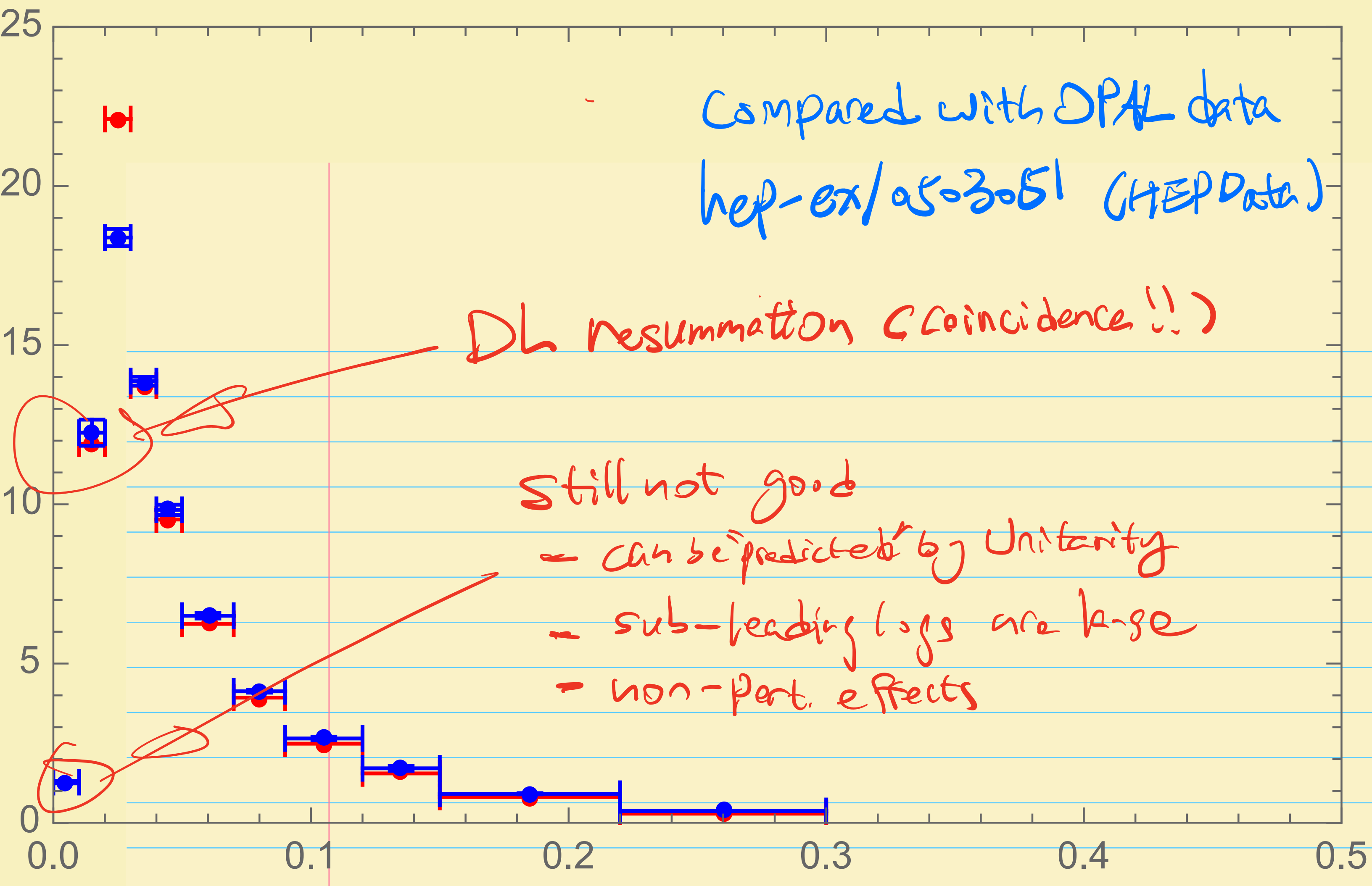
sub-leading in logs.

$$\Rightarrow \frac{\sigma(\delta)}{\sigma_{tot}} = e^{-\frac{\alpha_s}{\pi}C_F\log^2\delta} \quad + \text{ sub-leading terms} \ldots$$

$$= 1 - \frac{\alpha_s C_F}{\pi}\log^2\delta + \underbrace{\left(\frac{\alpha_s}{\pi}\right)^2 C_F^2\log^4\delta + \cdots}$$

reproduce the full NLO

predict higher orders



Compared with OPAL data
hep-ex/0503051 (HEPData)
DL resummation (coincidence!!)
Still not good
- can be "predicted" by Unitarity
- sub-leading logs are large
- non-pert. effects
25
20
15
10
5
0
0.0
0.1
0.2
0.3
0.4
0.5

Comment on the result:

- Can be improved to NLL by using the coherent branching to derive a RGE of the "Jet" Function

- DL = exp[NLO double log]

$$DL = \exp\left[\frac{2\alpha_s}{\pi}C_F\int_0^1\frac{d\theta^2}{\theta^2}\int_\tau^1\frac{dz}{z}\left\{\Theta(\delta-\tau)-1\right\}\right]$$

$$= 1 + \prod_{n\geq 1}^{\infty}\left[\frac{2\alpha_s}{\pi}C_F\int_0^1\frac{d\theta^2}{\theta^2}\int_\tau^1\frac{dz}{z}\left\{\Theta(\delta-\tau)-1\right\}\right]^n$$

with $\tau = z\theta^2/4$

Therefore at DL, we can replace

$$\Theta(\delta-\sum_i\tau_i) \longrightarrow \prod_{i=1}^{n}\Theta(\delta-\tau_i)$$

The emissions are all independent!

This holds in general for many observables