A short note on DIS

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DIS is among the simplest process but it is also tedious due to different form factor conventions introduced.

- all order form P. 8-9
- LO & NLO P. 30-32
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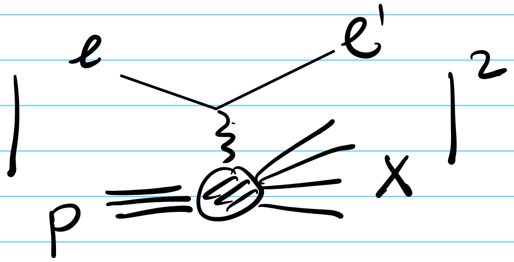
Unpolarized DIS. all order feature

We conside $P+\ell \to \ell'+X$



and define

$$s = (P+\ell)^2, \quad q = (\ell'-\ell)^2$$

To all orders. the x-sec is given by

$$d\sigma = \frac{1}{2s}\int \frac{d^3\ell'}{(2\pi)^3}\,\frac{1}{2\ell'}\,\underbrace{\frac{1}{2}\sum_{spin} \bar{u}\gamma^{\mu}u\,\bar{u}\gamma^{\nu}u}_{L^{\mu\nu}}$$

$$\times\, e^4 e_q^2 \left(\frac{1}{q^2}\right)^2 \int dPS_X\, \frac{1}{2}\langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle$$

$$\times\, (2\pi)^d\,\delta^{(d)}(q+P-P_X)$$

Here, we ignored higher order corrections from QED, and it is safe to put ℓ & ℓ' in 4-dimension.

① The leptonic tensor is

$$L^{\mu\nu} = \frac{1}{2}\sum_{\text{spin}} \bar{u}\gamma^{\mu}u\,\bar{u}\gamma^{\nu}u$$

$$= \frac{1}{2}\mathrm{Tr}[\not{\ell}\gamma^{\mu}\not{\ell}'\gamma^{\nu}]$$

$$= 2[\ell^{\mu}\ell'^{\nu} + \ell^{\nu}\ell'^{\mu} - g^{\mu\nu}\,\ell\cdot\ell']$$

(2) the lepton phase space can be written as

$$\int \frac{d^{d-1}\ell'}{(2\pi)^{d-1}}\frac{1}{2\ell'} = \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\frac{1}{16\pi^2}\,y\,\frac{dx_B}{x_B}\,dQ^2$$

Here, we have defined the DIS variables:

$$x_B \equiv -\frac{q^2}{2P\cdot q} = \frac{Q^2}{2P\cdot q}$$

$$y = \frac{2P\cdot q}{S} = \frac{Q^2}{x_B S}$$

The X-sec is then

$$\frac{d\sigma}{dx_B dQ^2} = \frac{1}{2s}\frac{y}{x_B}\frac{\alpha^2 e_q^2}{Q^4}\, 2\left[e^\mu e'^\nu + e^\nu e'^\mu - g^{\mu\nu} e\cdot e'\right]$$

$$W_{\mu\nu} \cdot \left\{ \begin{array}{l} \times \int dPS_X \, \frac{1}{2}\langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle \\ \times (2\pi)^d \delta^{(d)}(q+P-P_X) \end{array} \right.$$

Since we are interested in the unpolarized DIS, by gauge symmetry, the hadronic tensor can be written as

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1$$

$$+ \left(P^\mu - q^\mu \frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu \frac{P\cdot q}{q^2}\right) W_2$$

where,

$$g_{\mu\nu} W^{\mu\nu} = -(d-1) W_1 + \frac{y}{4x_B} S W_2$$

$$P^\mu P^\nu W_{\mu\nu} = -\frac{y}{4x_B} S W_1 + \frac{y^2}{16x_B^2} S^2 W_2$$

Solve For W_1 & W_2, we Find

$$W_1 = \frac{1}{d-2}\left(-g_{\mu\nu} + \frac{4x_B^2}{Q^2} p^\mu p^\nu\right) W_{\mu\nu}$$

$$W_2 = \frac{4x_B^2}{Q^2}\left(W_1 + \frac{4x_B^2}{Q^2} p^\mu p^\nu W_{\mu\nu}\right)$$

We note that

$$\boxed{-g_{\mu\nu} + \frac{4x_B^2}{Q^2} P_\mu P_\nu = \sum_\lambda \epsilon_{T\lambda}^{\mu} \epsilon_{T\lambda}^{*\nu}}$$

is nothing but the transversely polarized photon tensor. and

$$\boxed{\frac{4x_B^2}{Q^2} P_\mu P_\nu = \epsilon_L^\mu \epsilon_L^\nu}$$

is the longitudinal polarized photon tensor.

Contract $W_{\mu\nu}$ with $L^{\mu\nu}$, we find

$$L^{\mu\nu} W_{\mu\nu} (d-2)$$

$$= \left(1-y+\frac{1-\epsilon}{2}y^2\right) S^2 \sum_{\lambda} \epsilon_T^\mu \epsilon_T^\nu W_{\mu\nu} \frac{4x_B^2}{S}$$

$$+ (d-2)(1-y) S^2 \epsilon_L^\mu \epsilon_L^\nu W_{\mu\nu} \frac{4x_B^2}{S}$$

Hence, we find

$$\frac{d\sigma}{dx_B dQ^2} = \frac{\alpha^2 e_q^2}{Q^4} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{1}{1-\epsilon} \sum_{\lambda=L,T} f_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu W_{\mu\nu}$$

with $f_T = 1-y+\frac{1-\epsilon}{2}y^2$

$f_L = (d-2)(1-y)$

$$W_{\mu\nu} = \int dPS_X \frac{1}{2} |\langle P| J^\mu |X\rangle|^2 (2\pi)^d \delta^{(d)}(P+q-P_X)$$

We can also define the transverse and the longitudinal X-sec as

$$\sigma_T = \frac{1}{4\pi} \epsilon_T^{\mu} \epsilon_T^{\mu} W_{\mu\nu}$$

$$\sigma_L = \frac{1}{4\pi} \epsilon_L^{\mu} \epsilon_L^{\mu} W_{\mu\nu}$$

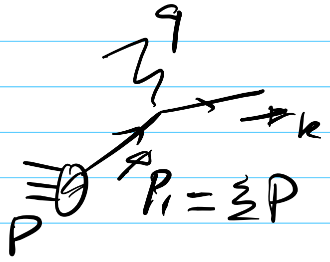
Note that our notation is related to F_2 & F_L in Zijlstra & van Neerven NPB 383 (1992) by

$$x_B^{-1} F_L/2 = \sigma_L$$

$$x_B^{-1} (F_2 - F_L) = \sigma_T$$

⊘ Leading Order :

at LO, $J^\mu = \bar{u}\gamma^\mu u$



Therefore we have

$$W^{(0)}_{\mu\nu} = \int \frac{d^d k}{(2\pi)^{d-1}} \underline{\delta(k^2)} \overset{(2\pi)}{(2\pi)^d} \delta^{(d)}(\xi P + q - k)$$

$$\times \; 2(k^\mu P_1^\nu + k^\nu P_1^\mu - g^{\mu\nu} P_1 \cdot k) \underset{\text{PDF}}{f(\xi)} \frac{d\xi}{\xi}$$

$$k^2 = (\xi P + q)^2 = 2\xi P\cdot q - Q^2$$
$$= 2\xi P\cdot q \left(1 - \frac{x_B}{\xi}\right)$$

PDF

$\frac{d\xi}{\xi}$: Count for the flux at the parton level

$$= \frac{2\pi}{2\xi P\cdot q} \delta\left(1 - \frac{x_B}{\xi}\right) f(\xi) \frac{d\xi}{\xi}$$

$$\times \; 2(k^\mu P_1^\nu + k^\nu P_1^\mu - g^{\mu\nu} P_1 \cdot k)$$

Now since

$$\sigma_T = \frac{1}{4\pi} \sum \epsilon_T^{\mu} \epsilon_T^{\nu} W_{\mu\nu}$$

$$= \frac{1}{4\pi} \frac{2\pi}{2\xi p\cdot q} \delta\left(1-\frac{x_B}{\xi}\right) F(\xi) \frac{d\xi}{\xi} 2(d-2)\xi p\cdot q$$

$$= \frac{1}{4\pi} (2\pi)(d-2) \delta\left(1-\frac{x_B}{\xi}\right) f(\xi) \frac{d\xi}{\xi}$$

$$\sigma_L = \frac{1}{4\pi} \epsilon_L^{\mu} \epsilon_L^{\nu} W_{\mu\nu} = 0$$

Therefore at LO

$$\frac{d\sigma}{dx_B dQ^2} = \frac{4\pi\alpha^2}{Q^4} e_q^2 \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} F_T \, \delta\left(1-\frac{x_B}{\xi}\right) F(\xi) \frac{d\xi}{\xi}$$

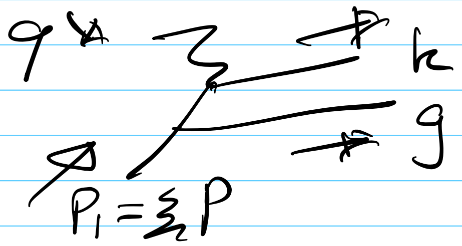
$$\hat{\sigma}_T^{(0)} = F_T \, \delta(1-z) \, , \quad \hat{\sigma}_L^{(0)} = 0$$

$$z = x_B/\xi$$

⊘ Next-to-Leading Order:

- real emission.

We start with the phase space for



$$\frac{d^d k}{(2\pi)^{d-1}} \frac{d^d g}{(2\pi)^{d-1}} \delta(k^2)\, \delta(g^2)\, \overset{(2\pi)}{(2\pi)^d}\, \delta^{(d)}(\xi p+q-k-g)$$

$$= \frac{2\pi}{(2\pi)^{d-1}} d^d g\, \delta(g^2)\, \delta((\xi p+q-k)^2)$$

$$= \frac{2\pi}{(2\pi)^{d-1}} \frac{1}{2\xi p\cdot q} d^d g\, \delta(g^2)\, \delta\left(1-\frac{x_B}{\xi}-\frac{2\xi p\cdot g+2q\cdot g}{2\xi p\cdot q}\right)$$

Since the phase space is Lorentz invariant. For simplicity, we choose the Breit-Frame, in which

$$P = \frac{\bar{n}\cdot P}{2} n^\mu . \quad q = \frac{\bar{n}\cdot q}{2} n^\mu + \frac{n\cdot q}{2} \bar{n}^\mu$$

aquire no transverse momentum,

Hence

$$2\xi P\cdot g = \xi \bar{n}\cdot P n\cdot g$$

$$2q\cdot g = \bar{n}\cdot q g^- + n\cdot q g^+$$

when

$$\bar{n} = (1,0,0,-1), \quad n = (1,0,0,1)$$

$$n\cdot g \equiv g^-, \quad \bar{n}\cdot g \equiv g^+.$$

It is then easy to find

$$1-\frac{x_B}{\xi}-\frac{2\xi p\cdot g+2q\cdot g}{2\xi p\cdot q}$$

$$=1-\frac{x_B}{\xi}-\left(1-\frac{x_B}{\xi}\right)\frac{g_T^2}{Q^2}\frac{1}{g^+/_{-q^+}}-\frac{x_B}{\xi}\frac{g^+}{-q^+}$$

The δ-function solves

$$g_T^2=\frac{1-z-z\,g^+/_{-q^+}}{1-z}\,Q^2\frac{g^+}{-q^+}$$

where $z=x_B/\xi$.

We consider

$$(\xi p - g)^2 = -2\xi p\cdot g \leq 0$$

$$\Rightarrow \xi\,\bar{n}\cdot p\; n\cdot g = \xi\,\bar{n}\cdot p\,\frac{g_T^2}{\bar{n}\cdot g}$$

$$= \xi\, 2p\cdot q\,\frac{(1-z-z\,g^+/-q^+)}{1-z} \leq 0$$

$$\Rightarrow z\,g^+/-q^+ \leq 1-z$$

if we let

$$\boxed{g^+/-q^+ = \frac{1-z}{z}\cdot t. \quad \text{then } t\leq 1}$$

Therefore we find for 2-emission

$$\int dPS_X (2\pi)^d \delta^{(d)}(P+q-P_X)$$

$$\boxed{\begin{aligned} &= \frac{2\pi}{(2\pi)^{d-1}} \frac{\Omega_{d-2}}{4} \left(\frac{1-z}{z}\right)^{-\epsilon} (Q^2)^{-\epsilon} dt\, t^{-\epsilon}(1-t)^{-\epsilon} \\ &\quad \times \frac{d\xi}{\xi} f(\xi) \end{aligned}}$$

We have

$$2P\cdot g = 2P\cdot q(1-t)$$

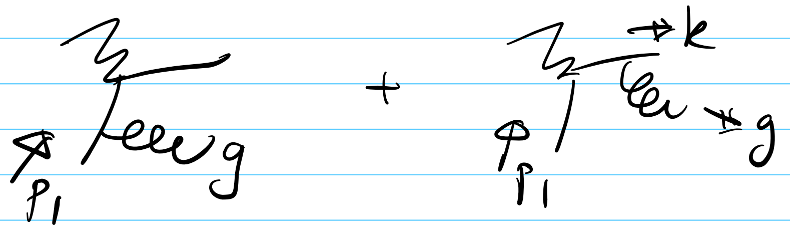
$$2k\cdot g = 2P\cdot q\,\xi(1-z)$$

$$2P\cdot k = 2P\cdot q\,t$$

$$\bar{n}\cdot g = (1-z)\bar{n}\cdot P\,t\cdot\xi$$

Now we calculate the matrix elements.

We start with the quark channel



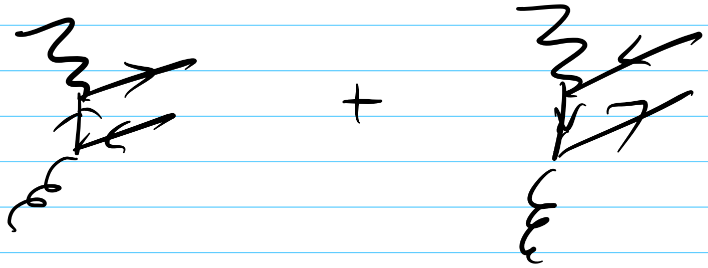
Bruteforce calculation gives

$$\epsilon_L^{\mu}\epsilon_L^{\nu}\circ\circ\circ = 4\pi\alpha_s C_F (d-2) 4 z t$$

$$-g_{\mu\nu}\circ\circ\circ$$

$$= 8\pi\alpha_s C_F (d-2)\times\left\{\left(\frac{1+z^2}{1-z}-\epsilon(1-z)\right)\frac{1}{1-t} - \frac{2z}{1-z}+\frac{1-t}{1-z}(1-\epsilon)+2\epsilon\right\}$$

The gluon channel is



$$\epsilon_L^{\mu}\epsilon_L^{\nu}\,\emptyset\emptyset\emptyset = \frac{1}{d-2}\,4\pi\alpha_s \;\; 16z(1-z)$$

$$-g_{\mu\nu}\,\emptyset\emptyset\emptyset = 16\pi\alpha_s(1-\epsilon)\,T_R\,\frac{1}{1-t}\,\frac{1}{t}$$

$$\times\left\{t^2+(1-t)^2-\frac{2z(1-z)}{1-\epsilon}-\frac{2\epsilon}{1-\epsilon}\,t(1-t)\right\}$$

We thus find

$$\sigma_{L,g}^{(1)} = 2\frac{\alpha_s}{2\pi} T_R \, 2z(1-z)$$

$$\sigma_{L,q}^{(1)} = \frac{\alpha_s}{2\pi} C_F \, z$$

for the longitudinal part.

While for the transverse part we have

$$\sigma^{(1)}_{T,g} = \frac{\alpha_s}{2\pi} T_R \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} (1-\epsilon) \left(\frac{1-z}{z}\right)^{-\epsilon} (Q^2)^{-\epsilon}$$

$$-g_{\mu\nu} \Big\{ \times \int dt\, t^{-\epsilon} (1-t)^{-\epsilon}$$

$$\times \left\{ \frac{t^2+(1-t)^2}{t(1-t)} - \frac{2z(1-z)}{1-\epsilon} [t(1-t)]^{-1} - \frac{2\epsilon}{1-\epsilon} \right\}$$

$$+ \underbrace{\frac{\alpha_s}{2\pi} T_R 4z(1-z)}_{p^\mu p^\nu}$$

Which gives

$$\sigma^{(1)}_{T,g} = \frac{\alpha_s}{2\pi} T_R \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} (1-\epsilon) \left(Q^2\right)^{-\epsilon}$$

$$\times 2 \left\{ -\frac{1}{\epsilon}\left(1-2z+2z^2\right) + \left(1-2z+2z^2\right) \ln\frac{1-z}{z} - 1 + 4z(1-z) \right\}$$

For the gluon channel

For the quark channel we have

$$G^{(1)}_{T,q,r} = \frac{\alpha_s}{2\pi} C_F \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{1-z}{z}\right)^{-\epsilon} (Q^2)^{-\epsilon}$$

$$\int_0^1 dt\, t^{-\epsilon}(1-t)^{-\epsilon}$$

$$\times\left\{\left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right)\frac{1}{1-t} - \frac{2z}{1-z} + \frac{1-t}{1-z}(1-\epsilon) + 2\epsilon\right\}$$

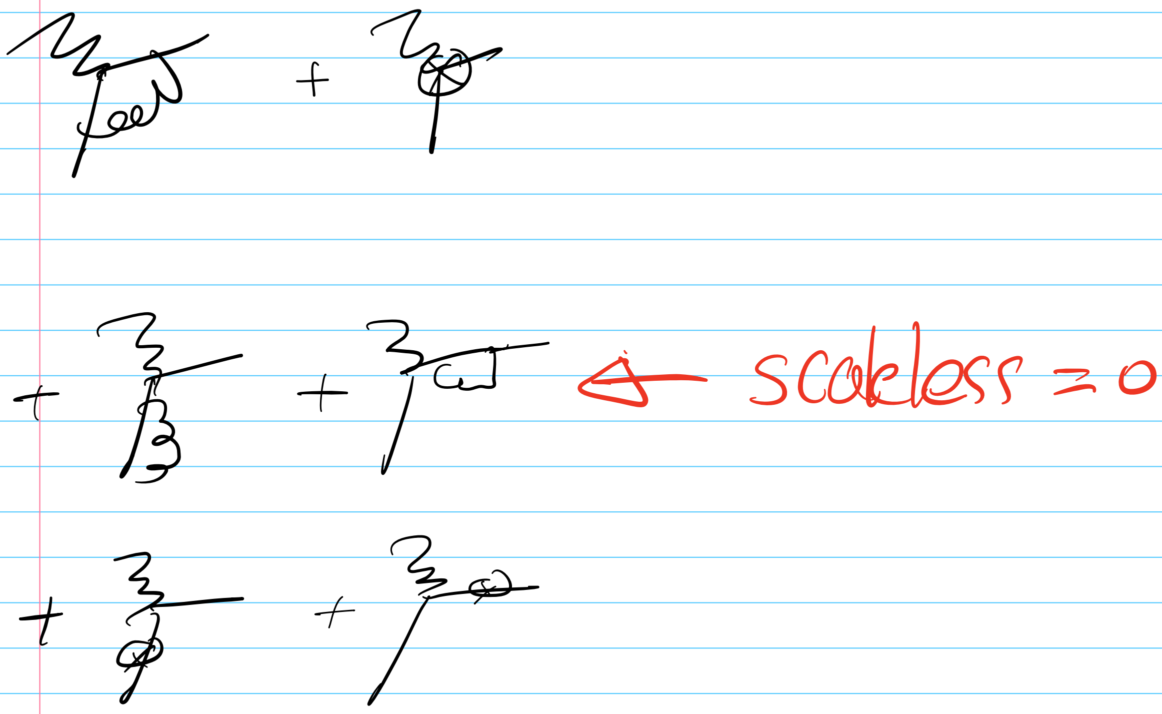
$$+ \frac{\alpha_s}{2\pi} C_F\, z$$

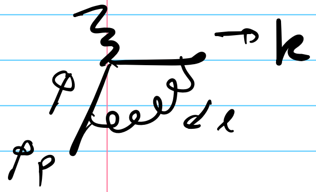
which gives

$$\sigma^{(1)}_{T,q,r} = \frac{\alpha_s}{2\pi} C_F \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} (1-\epsilon)(Q^2)^{-\epsilon}$$

$$\begin{aligned}
&\Big\{ \frac{2}{\epsilon^2}\delta(1-z) - \frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+} + \frac{3}{2\epsilon}\delta(1-z) \\
&+ (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{1+z^2}{1-z}\ln z - \frac{3}{2}\frac{1}{(1-z)_+} \\
&+ 3 - z + \left(\frac{7}{2} - \frac{\pi^2}{3}\right)\delta(1-z) \quad + z \Big\}
\end{aligned}$$

Now we add on the virtual correction & counter terms





define $[dl] \equiv \frac{d^d l}{(2\pi)^d}$

$$= \int [dl]\, \bar{u}_i\, i g_s \gamma^\alpha t^a \frac{i(\not{k}+\not{l})}{(k+l)^2} i e \gamma^\mu \frac{\not{p}+\not{l}}{(p+l)^2} i g_s t^a \gamma_\alpha u \frac{-i}{l^2}$$

$$= i e g_s^2 t^a t^a \int [dl] \boxed{\bar{u} \gamma^\alpha (\not{k}+\not{l}) \gamma^\mu (\not{l}+\not{p}) \gamma_\alpha u} \boxed{\frac{1}{(k+l)^2} \frac{1}{(p+l)^2} \frac{1}{l^2}}$$

Now we use the Feyn. Param. to find

$$\boxed{\frac{1}{(k+l)^2} \frac{1}{(p+l)^2} \frac{1}{l^2}} = \Gamma(3) \int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{[L^2+\alpha_2\alpha_3 q^2]^3}$$

$\hookrightarrow$

$$\alpha_1 l^2 + \alpha_2 l^2 + 2\alpha_2 k\cdot l + \alpha_3 l^2 + 2\alpha_3 l\cdot p$$

$$= l^2 + 2(\alpha_2 k + \alpha_3 p)\cdot l + (\alpha_2 k + \alpha_3 p)^2 - \alpha_2\alpha_3 2k\cdot p$$

$$= \underbrace{(l + \alpha_2 k + \alpha_3 p)}_{\equiv L}{}^2 + \alpha_2\alpha_3 q^2$$

$$(k-p)^2 = -2k\cdot p = q^2$$

now we simplify

$$\boxed{\bar{u}\gamma^{\alpha}(\not k+\not l)\gamma^{\mu}(\not l+\not p)\gamma_{\alpha}u}$$

$$= \bar{u}\gamma^{\alpha}(\not L+\bar{\alpha}_2\not k-\alpha_3\not p)\gamma^{\mu}(\not L-\alpha_2\not k+\bar{\alpha}_3\not p)\gamma_{\alpha}u$$

$$= -2\bar{u}(\not L-\cancelto{0}{\alpha_2\not k}+\bar{\alpha}_3\not p)\gamma^{\mu}(\not L+\bar{\alpha}_2\not k-\cancelto{0}{\alpha_3\not p})u$$

$$+2\epsilon\bar{u}(\not L+\cancelto{0}{\bar{\alpha}_2\not k}-\alpha_3\not p)\gamma^{\mu}(\not L-\alpha_2\not k+\cancelto{0}{\bar{\alpha}_3\not p})u$$

$$\longrightarrow -2\bar{u}\not L\gamma^{\mu}\not L u-2\bar{\alpha}_2\bar{\alpha}_3\bar{u}\not p\gamma^{\mu}\not k u$$

$$+2\epsilon\bar{u}\not L\gamma^{\mu}\not L u+2\epsilon\alpha_2\alpha_3\bar{u}\not p\gamma^{\mu}\not k u$$

$$= -2(1-\epsilon)\bar{u}\not L\gamma^{\mu}\not L u$$

$$-(2\bar{\alpha}_2\bar{\alpha}_3-2\epsilon\alpha_2\alpha_3)\bar{u}\not p\gamma^{\mu}\not k u$$

$$\rightarrow \left\{ \frac{(d-2)^2}{d} L^2 + 2(\bar{\alpha}_2\bar{\alpha}_3 - \epsilon\alpha_2\alpha_3)(-q^2) \right\} \bar{u}\gamma^\mu u$$

Now we use

$$① \quad \int [dL] \frac{L^2}{(L^2+\alpha_2\alpha_3 q^2)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (\alpha_2\alpha_3)^{-\epsilon} (-q^2)^{-\epsilon}$$

$$② \quad \int [dl] \frac{1}{(l^2+\alpha_2\alpha_3 q^2)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} (\alpha_2\alpha_3)^{-1-\epsilon} (-q^2)^{-1-\epsilon}$$

to find

$$\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \left[\frac{1}{2\epsilon_{UV}} + \frac{1}{2} - \frac{1}{\epsilon^2} - \frac{2}{\epsilon} - \frac{9}{2}\right]$$

Add up all terms

$$\Delta^{\text{Virt.}}_{q,T} = \Delta^{(0)} \cdot$$

$$\frac{\alpha_s}{2\pi} C_F \left[\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \times \left\{ -\frac{2}{\epsilon^2} - 3\frac{1}{\epsilon} - 8 \right\}$$

Immediately we find

$$\sigma^{(1)}_{q,T} = \frac{\alpha_s}{2\pi} C_F \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} (1-\epsilon)(Q^2)^{-\epsilon}$$

$$\Big\{ -\frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+} - \frac{3}{2\epsilon}\delta(1-z)$$

$$+ (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{1+z^2}{1-z}\ln z - \frac{3}{2}\frac{1}{(1-z)_+}$$

$$+ 3 - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z) \Big\}$$

Summary:

$$\hat{\sigma} = \frac{4\pi\alpha^2}{Q^4} \sum_{i=-N_f}^{N_f} e_{q_i}^2 \, f_\lambda \, \hat{\sigma}_{q_i,\lambda}$$

$$+ \frac{4\pi\alpha^2}{Q^4} \sum_{i=1}^{N_f} e_{q_i}^2 \, f_\lambda \, \hat{\sigma}_{g,\lambda}$$

With $\hat{\sigma} = \hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \hat{\sigma}^{(1)}$

$f_L = 2-2y$, $f_T = 1-y+y^2/2$

LO:

$$\sigma^{(0)}_{q,T} = \delta(1-z).$$

$$\sigma^{(0)}_{q,L} = \sigma^{(0)}_{g,L} = \sigma^{(0)}_{g,T} = 0$$

NLO:

$$\hat{\sigma}^{(1)}_{g,L} = 2T_R\, 2z(1-z)$$ $q, \bar{q}$ included

$$\hat{\sigma}^{(1)}_{q,L} = C_F\, z$$

and

$$\hat{\sigma}^{(1)}_{g,T} = 2T_R \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}(1-\epsilon)\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}$$

$$\times\left\{-\frac{1}{\epsilon}(1-2z+2z^2)+(1-2z+2z^2)\ln\frac{1-z}{z} - 1 + 4z(1-z)\right\}$$

$q, \bar{q}$ included

$$\hat{\sigma}^{(1)}_{q,T} = C_F \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}(1-\epsilon)\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}$$

$$\times\left\{-\frac{1}{\epsilon}\left(\frac{1+z^2}{1-z}\right)_+ + \frac{1+z^2}{1-z}\ln\frac{1-z}{z} - \frac{3}{2}\frac{1}{1-z}\right.$$

$$\left. + 3 - \left(\frac{9}{2} + \frac{\pi^2}{3}\right)\delta(1-z)\right\}$$

+-distribution for $z \to 1$ divergence is understood.

Flavor Singlet & non-singlet

We can further write the DIS Xsec as

$$d\sigma = d\sigma_q + d\sigma_g$$

where

$$d\sigma_q = \sum_{i=-N_F}^{N_F} \frac{4\pi\alpha}{Q^4} e_{q_i}^2 \hat{\sigma}_q F_{q_i/p}$$

&

$$d\sigma_g = \sum_{i=-N_F}^{N_F} \frac{4\pi\alpha}{Q^4} e_{q_i}^2 \frac{\hat{\sigma}_g}{2} F_{g/p}$$

↳ only q or $\bar{q}$

where $\hat{\sigma}_q$ & $\hat{\sigma}_g$ is given before and are both i-independent, Now we introduce the Flavor Singlet distribution

$$F_q^S = \sum_{i=1}^{N_F} f_{\bar{q}_i} + f_{q_i}$$

$$F_g^S = f_g$$

and the non-singlet distribution

$$F_i^{NS} = N_F \left(f_{q_i} + f_{\bar{q}_i} \right) - F_q^S$$

The X-section can then be written as

$$d\sigma = \frac{1}{N_F} \sum_{i=1}^{N_F} \frac{4\pi\alpha^2}{Q^4} e_{q_i}^2 \hat{G}_q \, F_i^{NS}$$

$$+ \frac{1}{N_F} \sum_{i=1}^{N_F} \frac{4\pi\alpha^2}{Q^4} e_{q_i}^2 \hat{G}_q \, F_q^{S}$$

$$+ \frac{1}{N_F} \sum_{i=1}^{N_F} \frac{4\pi\alpha^2}{Q^4} e_{q_i}^2 N_F \hat{G}_g \, F_g^{S}$$

Now we study the evolution for F_i^{NS} & $F^{S}_{q/g}$.

Note that

$$\frac{d f_{q_i}}{d \ln \mu^2} = \left(P^{V}_{qq} \delta_{ij} + P^{S}_{qq} \right) f_{q_j} + \left(P^{V}_{q\bar{q}} \delta_{ij} + P^{S}_{q\bar{q}} \right) f_{\bar{q}_j} + P_{qg} f_g$$

&

$$\frac{dF_{\bar{q}_i}}{d\ln\mu^2} = \left(P^V_{qq}\delta_{ij} + P^S_{qq}\right) F_{\bar{q}_j} + \left(P^V_{q\bar{q}}\delta_{ij} + P^S_{q\bar{q}}\right) f_{q_j} + P_{qg} F_g$$

where we have used the charge and Flavor Symmetry,

$$P_{q\bar{q}} = P_{\bar{q}q}\,, \quad P_{qq} = P_{\bar{q}\bar{q}}$$

Therefore

$$\frac{d}{d\ln\mu^2}\left(f_{q_i}+f_{\bar{q}_i}\right)$$

$$= \sum_{j=1}^{N_F}\left(P^V_{qq}\,\delta_{ij} + P^S_{qq}\right)\left(f_{q_j}+f_{\bar{q}_j}\right)$$

$$+ \sum_{j=1}^{N_F}\left(P^V_{q\bar{q}}\,\delta_{ij} + P^S_{q\bar{q}}\right)\left(f_{q_j}+f_{\bar{q}_j}\right)$$

$$+ 2P_{qg}\,f_g$$

and therefore for singlet

$$\frac{d}{d\ln\mu^2} F_q^S$$

$$= \Big[\underbrace{P^V_{qq} + P^V_{q\bar{q}}}_{P_{NS}^{+}} + N_F \underbrace{\left(P^S_{qq} + P^S_{q\bar{q}}\right)}_{P_{ps}} \Big] F_q^S$$

$$+ 2N_F P_{qg} F_g$$

$$\equiv P^S_{qq} F^S_q + P^S_{qg} F^S_g$$

$$\frac{d}{d\ln\mu^2} F_g^S = P_{gg} F_g^S + P_{gq} \sum_i f_{q_i} + f_{\bar{q}_i}$$

$$\equiv P_{gg}^S F_g^S + P_{gq}^S F_q^S$$

⇒ For Flavor Singlet

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} F_q^S \\ F_g^S \end{pmatrix} = \begin{pmatrix} P_{qq}^S & \overset{\nearrow 2N_F P_{qg}}{P_{qg}^S} \\ \underset{\overset{\uparrow}{P_{qg}}}{P_{gq}^S} & \underset{\overset{\uparrow}{P_{gg}}}{P_{gg}^S} \end{pmatrix} \begin{pmatrix} F_q^S \\ F_g^S \end{pmatrix}$$

For non-singlet we have

$$\frac{d}{d\ln\mu^2} F_{q_i}^{NS}$$

$$= \left(P_{qq}^{V} + P_{q\bar{q}}^{V}\right) F_{q_i}^{NS}$$

$$\equiv \underset{\uparrow\ \text{Flavor Independent}}{P_q^{NS}}\ F_{q_i}^{NS}$$

Summary :

$$d\sigma = \frac{1}{N_F} \sum_{i=1}^{N_F} \frac{4\pi\alpha^2}{Q^4} e_{q_i}^2 \hat{\sigma}_q F_i^{NS}$$

$$+ \frac{1}{N_F} \sum_{i=1}^{N_F} \frac{4\pi\alpha^2}{Q^4} e_{q_i}^2 \hat{\sigma}_q F_q^S$$

$$+ \frac{1}{N_F} \sum_{i=1}^{N_F} \frac{4\pi\alpha^2}{Q^4} e_{q_i}^2 N_F \hat{\sigma}_g F_g^S$$

where

$$F_i^{NS} = N_F \left(F_{q_i} + F_{\bar{q}_i}\right) - F_q^S$$

$$F_q^S = \sum_i^{N_F} f_{q_i} + f_{\bar{q}_i}$$

$$F_g^S = f_g$$

and satisfy

$$\frac{d}{d\ln\mu^2} F_i^{NS} = P_q^{NS} \; F_i^{NS}$$

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} P_q^S \\ P_g^S \end{pmatrix} = \begin{pmatrix} P_{qq}^S & P_{qg}^S \\ P_{gq}^S & P_{gs}^S \end{pmatrix}\begin{pmatrix} P_q^S \\ P_g^S \end{pmatrix}$$

$$P_{qg}^S = 2N_F \, P_{qg}$$

$$P_{gq}^S = P_{gq} \; , \; P_{gg}^S = P_{gg}$$